1 Basics

Definition 1 (Zorn's lemma) *If every totall ordered set in a poset P has an upper bound in P, then P contains a maximal element.*

Definition 2 *An element m of a poset P is called **maximal** if there is no $y \in P$ such that $m \leq y$ and $m \neq y$. That is, if m is maximal and $m \leq x$ implies $m = x$*

Definition 3 *Let V be a non-empty set and $\mathbb{F}$ be a field. Then, V is a **vector space** over $\mathbb{F}$ if for $+ : V \times V \longrightarrow V$ and $. : \mathbb{F} \times V \longrightarrow V$ if*

1. $\mathbf{u} + \mathbf{v} \in V$ *, as implied by the definition of the function +*
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
4. *There is an object $\mathbf{0}$ in V, called a zero vector for V, such that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}\ \forall \mathbf{u}$ in V.*
5. *For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a negative of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$*
6. *If α is any scalar, i.e. an element of $\mathbb{F}$, and $\mathbf{u}$ is any object in V, then $\alpha\mathbf{u}$ is in V, as implied by the definition of scalar multiplication function.*
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$
8. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$
9. $(\alpha\beta)\,\mathbf{u} = \alpha(\beta\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in V$ *and* $1 \in \mathbb{F}$

Definition 4 *Let X and Y be vector spaces over the same field. Then, an operator $T : X \longrightarrow Y$ is **linear** if for all $x, y \in X$ and scalars α, $T(x + y) = T(x) + T(y)$ and $T(\alpha x) = \alpha T(x)$.*

Lemma 5 *An operator $T : X \longrightarrow Y$ is linear if and only if $\forall x, y \in X$ and $\forall \alpha, \beta \in F$, $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$*

Proof. $T(\alpha x + \beta y) = T(\alpha x) + T(\beta y) = \alpha T(x) + \beta T(y)$

Conversely, by definition, $T(\alpha x) = \alpha(T(x)) = \alpha T(x)$

For the first property of linearity, we have

$$\begin{aligned} & T(\alpha x + \beta y) \\ = \ & \alpha T(x) + \beta T(y) \\ = \ & T(\alpha x) + T(\beta y) \end{aligned}$$

If $\alpha x = u$ and $\beta y = v$, then from $T(\alpha x + \beta y) = T(\alpha x) + T(\beta y)$, we have $T(v + u) = T(u) + T(v)$ ■

Of special importance is the fact that for any linear operator T and $\mathbf{0}$ vector, $T(\mathbf{0}) = \mathbf{0}$

Proof. $T(\mathbf{0}) = T(x - x)$
$= T(x + (-1x))$
$= T(x) + T(-1x)$
$= T(x) - T(x)$
$=\mathbf{0}$ ■

Definition 6 *Let Let X and Y be vector spaces and $T : X \longrightarrow Y$ be an operator. Then, the* ***null space*** *$\mathcal{N}(T)$ or kernel of T, denoted by $\ker T$ is the set $\{x \mid T(x) = 0\}$.*

This is the complement of the supp$T$

Theorem 7 *Let T be a linear operator. Then $\ker T$ is a vector space.*

Proof. For $x, y \in \ker T$, $T(\alpha x - \beta y) = \alpha T(x) - \beta T(y) = 0$ so that $\alpha x - \beta y \in \ker T$ ■

Definition 8 *Let N be a vector space over a field $\mathbb{F}$. A* ***norm*** *on N is a real-valued function $\|.\| : N \longrightarrow [0, \infty)$ such that*

N1 $\|x\| > 0$ $\forall x \in N$ *and* $\|x\| = 0 \iff x = 0$

N2 $\|\alpha x\| = |\alpha| \, \|x\|, \forall \ \alpha \in \mathbb{F}, x \in N$ *(homogeneity)*

N3 $\|x + y\| \leq \|x\| + \|y\|$ *for arbitrary $x, y \in N$ (triangle inequality).*

Theorem 9 *N is Banach if and only if every absolutely convergent series is convergent.*

Proof. If B is a Banach space, then let $\sum \|x_k\|$ be convergent for a sequence x_k. What this means is that we can have an integer N such that

$$\sum_{k=N}^{\infty} \|x_k\| < \epsilon$$

If we have an N such that for $n, m > N$, the partial sums S_n and S_m for

$$S_n = \sum_{k=1}^{n} x_k$$

can give us

$$\|S_n - S_m\| = \left\| \sum_{k=n+1}^{m} x_k \right\|$$

for $n < m$. Then,

$$\left\|\sum_{k=n+1}^{m} x_k\right\| \leq \sum_{k=n+1}^{m} \|x_k\| < \epsilon$$

i.e. $\|S_n - S_m\| < \epsilon$ is Cauchy. Since this is a Cauchy sequence, we must have a limit point S. Thus, $S = \lim_{n\to\infty} S_n$ exists and is finite so that the series converges.

Conversely, let every absolutely convergent series be convergent and let x_n be a Cauchy sequence. Since $\|x_n - x_m\| < \epsilon$, we can have an integer n_1 so that $\|x_n - x_m\| < 2^{-1}$ for $n, m \geq n_1$. Again, we can find a n_2 such that

$$\|x_n - x_m\| < 2^{-2}$$

for $n, m \geq n_2$. Moving on, we can find n_k such that

$$\|x_n - x_m\| < 2^{-k}$$

for $n, m \geq n_k$. In particular, since

$$n_{k+1} > n_k \geq n_k$$

we can have $n = n_k$ and $m = n_{k+1}$ so that we have

$$\left\|x_{n_{k+1}} - x_{n_k}\right\| < 2^{-k}$$

Thus, we can have a subsequence n_k such that

$$\left\|x_{n_{k+1}} - x_{n_k}\right\| < 2^{-k}$$

If we substitute y_k for $x_{n_{k+1}} - x_{n_k}$, then

$$\sum \|y_k\| < \epsilon$$

implying that we have an absolutely convergent series. By our hypothesis, it should converge. Thus, $\sum y_k \longrightarrow S$ and the sequence of partial sums of y_k converges and this is a subsequence of x_n. Since x_n has a convergent subsequence and is Cauchy, it will also converge to the same limit as its subsequence. Thus, this particular Cauchy sequence converges, which implies our space is Banach. ■

Theorem 10 *Every finite dimensional normed space N is complete.*

Proof. Consider the Cauchy sequence x_k and a set of linearly independent basis $\{e_1, e_2, ..., e_n\}$. We can represent the k-th term of this sequence as $x_k = \alpha_1^{(k)} e_1 + \alpha_2^{(k)} e_2 + ... + \alpha_n^{(k)} e_n$ where the superscript is not a power but rather serves as a reminder that the scalars α_i will depend on k. Now, For $n, m > N$

$$\begin{aligned} \|x_n - x_m\| &< c\epsilon \\ \Longrightarrow &\left\|\sum_{j=1}^{n} \left(\alpha_j^{(n)} - \alpha_j^{(m)}\right) e_j\right\| < \epsilon \end{aligned}$$

$\implies \exists c$ such that $c\left|\sum_{j=1}^{n}\alpha_j^{(n)}-\alpha_j^{(m)}\right| \leq \left\|\sum_{j=1}^{n}\left(\alpha_j^{(n)}-\alpha_j^{(m)}\right)e_j\right\| < c\epsilon$

i.e. $\left|\sum_{j=1}^{n}\left(\alpha_j^{(n)}-\alpha_j^{(m)}\right)\right| < \epsilon$ which is a Cauchy sequence of scalars belonging to a complete field. Needless to say, we have convergence so that we can use n such limits of the form α_i to construct $x = \alpha_1 e_1 + \alpha_2 e_2 + ... + \alpha_n e_n$ so that $x \in N$. Now,

$$
\begin{aligned}
\|x_k - x\| &= \left\|\sum_{j=1}^{n}\left(\alpha_j^{(k)}-\alpha_j\right)e_j\right\| \\
&\leq \left|\sum_{j=1}^{n}\left(\alpha_j^{(k)}-\alpha_j\right)\right| \|e_j\| \\
&\leq b\left|\sum_{j=1}^{n}\left(\alpha_j^{(k)}-\alpha_j\right)\right|
\end{aligned}
$$

where $b = \max_j e_j$

$$
\begin{aligned}
\|x_k - x\| &\leq b\left|\sum_{j=1}^{n}\left(\alpha_j^{(k)}-\alpha_j\right)\right| \\
&< b\epsilon
\end{aligned}
$$

so that the Cauchy sequence converges, implying convergence. ■

Definition 11 *Let $(N, \|.\|_1)$ and $(M, \|.\|_2)$ be normed spaces and $T : N \longrightarrow M$ a linear operator. The operator T is said to be* ***bounded*** *if there is a real number $c > 0$ such that $\forall x \in N$ $\|T(x)\|_2 \leq c\|x\|_1$*

Theorem 12 *Let T be a linear operator. Then, T is continuous if and only if it is bounded.*

Proof. Let $T : X \longrightarrow Y$ be continuous. Then, $\|T(\mathbf{x}) - T(\mathbf{x}_0)\|_Y < \varepsilon$ whenever $\|\mathbf{x} - \mathbf{x}_0\|_X < \delta$

or $\|T(\mathbf{x} - \mathbf{x}_0)\|_Y < \varepsilon$ whenever $\|\mathbf{x} - \mathbf{x}_0\|_X < \delta$

Let $\mathbf{x} - \mathbf{x}_0 = \frac{\varepsilon\mathbf{y}}{a\|\mathbf{y}\|_X}$ for $a > 0$. This is justified since the denominator is bounded and not equal to zero. Then,

$$\left\|T(\frac{\varepsilon\mathbf{y}}{c\|\mathbf{y}\|_X})\right\|_Y < \varepsilon \Rightarrow \|T(\mathbf{y})\|_Y < a\|\mathbf{y}\|_X$$

or $\|T(\mathbf{y})\|_Y \leq c\|\mathbf{y}\|_X$ for some $0 < c < a$

Conversely, $\|T(\mathbf{y})\|_Y \leq c\|\mathbf{y}\|_X$

Let $\mathbf{y} = \mathbf{x} - \mathbf{x}_0$ for $\|\mathbf{y}\|_X = \|\mathbf{x} - \mathbf{x}_0\|_X < \delta$

Then, $\|T(\mathbf{x}) - T(\mathbf{x}_0)\|_Y < c\delta = \varepsilon$ whenever $\|\mathbf{x} - \mathbf{x}_0\|_X < \delta$ ■

Corollary 13 *Let $T : N \longrightarrow M$ be a linear operator and N, M are normed spaces. Then, if T is continuous at a single point, then it is continuous.*

Corollary 14 $x_n \longrightarrow x$ *implies* $T(x_n) \longrightarrow T(x)$

Proof. Let $\|(x_n - x)\| < \epsilon / \|T\|$ for $n > N$. Then,

$$\begin{aligned} & \|T(x_n) - T(x)\| \\ = & \|T(x_n - x)\| \\ \leq & \|T\| \|(x_n - x)\| \\ < & \epsilon \end{aligned}$$

■

Theorem 15 $\ker T$ *is closed for linear, bounded T.*

Proof. For a limit point x of $\ker T$, there exists a sequence $x_n \longrightarrow x$. From this, $T(x_n) \longrightarrow T(x)$. Since $T(x_n) = 0$, then $T(x) = 0$ so that $x \in \ker T$ ■

This goes out to say that the range of a bounded operator need not be closed. This enables us to differentiate between bounded operators and compact operators. Since the operator is continuous, the inverse images of open (resp. closed) sets must be open (resp. closed). If the image of a subset of the range is not closed, then the domain must necessarily not be closed.

All the corresponding results hold for functionals.

We have already seen that finite dimensional spaces are much simpler than infinite dimensional ones in certain aspects. Of particular note is the role of operators and functionals on such spaces. We will show this by incorporating matrices into our discussions. For a review of matrices, see the appendix.

Recall that for an n-dimensional vector, an $r \times n$ matrix acts on it to give a r-dimensional vector. Thus, linear operators on finite dimensional spaces can be viewed as matrices. Matrix operation is associative, linear and in some cases, bounded and invertible, making it a perfect candidate for our present discussion. We also have the added advantage of going computational. Here's how the equivalence can be made:

Let $T : X \longrightarrow Y$ be a linear operator with X, Y normed spaces:

Let $\dim X = n < \infty$ and $\dim Y = r < \infty$ and let basis of X be $e_1, e_2, ..., e_n$. Then, every vector x in the domain can be represented using scalars α_i's such that

$$x = \sum_{i=1}^{n} \alpha_i e_i$$

Applying the linear operator, we get

$$y = T(x) = \sum_{i=1}^{n} \alpha_i T(e_i)$$

If $\bar{e}_1, \bar{e}_2, ..., \bar{e}_r$ are the basis of the range, then every vector y can be represented as

$$y = \sum_{i=1}^{r} \beta_i \bar{e}_i$$

Now, every $T(e_k)$ is a vector in the range. Hence, this, too can be represented as

$$T(e_k) = \sum_{i=1}^{r} \gamma_{ik} \bar{e}_i$$

where β_i, γ_i are scalars in the field of the codomain. The scalar γ will vary, depending on the vector $T(e_k)$, which justifies the subscript. Now, the two representations of y should agree. That is,

$$y = \sum_{k=1}^{r} \beta_k \bar{e}_k = \sum_{i=1}^{n} \alpha_k T(e_k)$$

This equation implicitly assumes that we can know the (unique) images of each member of the basis of the domain. The representation of the vector $T(e_k)$ is placed into this equation to give

$$\begin{aligned} y &= \sum_{i=1}^{n} \alpha_k T(e_k) = \sum_{i=1}^{n} \alpha_k \sum_{i=1}^{r} \gamma_{ik} \bar{e}_i \\ &= \sum_{i=1}^{r} \sum_{i=1}^{n} (\alpha_k \gamma_{ik}) \bar{e}_i \end{aligned}$$

Now, y cannot have two different representations. We must have $\beta_i = \sum_{i=1}^{n} (\alpha_k \gamma_{ik})$ for each i. This should look familiar: it is a tuple of a vector b if you perform the matrix multiplication $Ax = b = (\beta_i)$. Now, if we can determine these β_i's, we know the value of $T(x)$. By now, it should be clear that in order to determine the matrix equivalent A of T, we can safely say that $A = (a_{ik}) = (\gamma_{ik})$. Notice that this depends on the choice of basis for the domain so that there can be many different matrices by changing the choice of basis of the domain.

Note that this is valid only for finite dimensional spaces!

Let us do an example to hit the point home.

Example 16 *Let's say we have an operator that skews a vector and reduces a dimension. That is,* $T(x, y, z) = (3x, 2y)$. *To make our lives simple, we will assume* $e_1 = (1, 0, 0)$ *and* $e_2 = (0, 1, 0)$ *as our basis in both spaces with an addition of* $e_3 = (0, 0, 1)$ *in the domain. Now,* $T(e_1) = (3, 0)$, $T(e_2) = (0, 2)$ *and* $T(e_3) = (0, 0)$. *Therefore, the corresponding matrix is*

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

so that

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x \\ 2y \end{bmatrix}$$

Recall the definition of supremum. It is an upper bound and the lowest of the upper bounds of a set. Thus, we can collect all x such that $\frac{\|T(x)\|}{\|x\|} \leq c$ and define a supremum out of it. If we can find a smallest such c, then we have

Definition 17 *The* ***norm*** *of a bounded linear operator T, denoted by $\|T\|$, is defined as $\|T\| = \sup\limits_{x} \frac{\|T(x)\|}{\|x\|}$.*

Needless to say, this is valid when $\|x\| \neq 0$. Also, the norm of $\|T\|$ can be taken over normalised vectors so that $\|T\| = \sup\limits_{\|x\|=1} \|T(x)\|$.

Note the requirement for a norm to exist: the operator must be bounded.

Since $\|T\| = \sup\limits_{x} \frac{\|T(x)\|}{\|x\|}$, we can safely say that $\|T\| \geq \frac{\|T(x)\|}{\|x\|}$ for all x so that we have for ourselves the inequality

$$\|T(x)\| \leq \|T\| \|x\|$$

Thus, the following definitions are equivalent:

$\|T\| := \sup\limits_{x\neq 0} \frac{\|T(x)\|}{\|x\|} = \sup\limits_{\|x\|=1} \|T(x)\| = \sup\limits_{0<\|x\|\leq 1} \|T(x)\| = \sup\limits_{0<\|x\|<1} \|T(x)\| = \inf\{k : \|T(x)\| \leq k\|x\|, \forall x\}$

Proof. Let $A = \left\{ \frac{\|T(x)\|}{\|x\|} : x \in X\setminus\{0\} \right\}$

$B = \{\|T(x)\| : x \in X\setminus\{0\} \text{ and } \|x\| = 1\}$

$C = \{\|T(x)\| : x \in X\setminus\{0\} \text{ and } \|x\| \leq 1\}$

$D = \{\|T(x)\| : x \in X\setminus\{0\} \text{ and } \|x\| < 1\}$

Since equal sets have the same supremum, we will show that $A = B = C = D$

Clearly, A contains B, C and D.

Let $a \in A$

$\iff a = \frac{\|T(x)\|}{\|x\|}$ for some $x \in X\setminus\{0\}$

Since X is a norm space and closed under scalar multiplication, we can let $y\|x\| = x$

$\iff y \neq 0$ and $\|y\| = 1$ so that $a = \|T(y)\|$ for some $y \in X\setminus\{0\}$

$\iff a \in B$

$\iff A = B$

It is clear that $D \subseteq C$ and that $B \cup D = C$ so that $B \subseteq C$ as well.

Further, $B = A \subseteq C$ so that we have $B = A = C$

To show that $B \subseteq D$

$a \in B$

$\implies a = \|T(x)\|$ for some $x \in X\setminus\{0\}$ and $\|x\| = 1$

Assume that $\exists x_n$ such that $x_n \longrightarrow x$.

Let $y_n = \frac{n-1}{n} x_n$. Then, $y_n \longrightarrow y$. Furthermore, $\|y_n\| < \|x_n\|$ so that $\|x\| = 1$ implies $\|y\| < 1$

Then, $a_n = \|T(y_n)\| = \left|\frac{n-1}{n}\right| \|T(x_n)\| \longrightarrow \|T(y)\| = a$

Finally, we show that $\sup\limits_{x\neq 0} \frac{\|T(x)\|}{\|x\|} = \inf\{k : \|T(x)\| \leq k\|x\|, \forall x\}$

Assume that $\sup\limits_{x\neq 0} \frac{\|T(x)\|}{\|x\|} = \alpha$

Then, $\|T(x)\| \leq \alpha \|x\|$

$\implies \sup\limits_{x\neq 0} \frac{\|T(x)\|}{\|x\|} = \alpha \geq \inf\{k : \|T(x)\| \leq k\|x\|, \forall x\}$

Next, $\inf\{k : \|T(x)\| \leq k\|x\|, \forall x\} \geq \frac{\|T(x)\|}{\|x\|} \geq \alpha - \frac{1}{n}$ for all n

So that $\inf\{k : \|T(x)\| \leq k\|x\|, \forall x\} = \alpha = \sup\limits_{x\neq 0} \frac{\|T(x)\|}{\|x\|}$ ■

This norm satisfies all the conditions of a norm space:

Proof. For N1, $\|T\| \geq \frac{\|T(x)\|}{\|x\|} \geq 0$. Next, $\|T\| = 0$ if and only if $\sup \frac{\|T(x)\|}{\|x\|} = 0$ which implies $\sup \|T(x)\| = 0$. Since we have a supremum of non-negative numbers and this is equal to zero, therefore $\|T\| = 0$ if and only if $\|T(x)\| = 0$ for all x. This is only possible when T is the zero operator.

For N2, $\|\alpha T\| = \sup\limits_{x} \frac{\|\alpha T(x)\|}{\|x\|} = \sup\limits_{x} \frac{|\alpha|\|T(x)\|}{\|x\|} = |\alpha| \sup\limits_{x} \frac{\|T(x)\|}{\|x\|} = |\alpha| \|T\|$. In the second step, the homogenity property is applied because of the norm of $\mathcal{R}(T)$. In the third step, the scalar can be factored out because it has no role in the supremum since it does not depend on x.

For N3, $\sup \|(T_1 + T_2)(x)\| = \sup \|T_1(x) + T_2(x)\|$. Since $\|T_1(x) + T_2(x)\| \leq \|T_1(x)\| + \|T_2(x)\|$ and so also their supremum, thus $\sup \|T_1(x) + T_2(x)\| \leq \sup \|T_1(x)\| + \sup \|T_2(x)\|$

From this, we can have $\|T_1 + T_2\| \leq \|T_1\| + \|T_2\|$ ■

Thus, the space of bounded, linear operators $T : X \longrightarrow Y$ between vector spaces X and Y, denoted by $B(X, Y)$ is a norm space. Under point-wise addition and scalar multiplication, $B(X, Y)$ is a vector space.

Theorem 18 *If Y is a Banach space, then so is $B(X, Y)$*

Proof. Now, remember, elements of $B(X, Y)$ are linear operators T so that if we want to show that an arbitrary Cauchy sequence in $B(X, Y)$ converges, we must take a sequence of operators and show that it converges. Let (T_n) be a Cauchy sequence of operators in $B(X, Y)$. Thus, by definition, for all $\epsilon > 0$, $\exists N$ such that $\|T_n - T_m\| < \epsilon$ whenever $n, m > N$. For all $x \in X$ and $n, m > N$, we have $\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$ (point-wise addition) $\leq \|T_n - T_m\| \|x\|$ (T_i's are bounded)

Therefore, $\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\| \|x\| < \epsilon \|x\|$. Now, for any fixed x and given ϵ', we may choose $\epsilon = \epsilon_x$ such that $\epsilon_x \|x\| < \epsilon'$. Then, $\|T_n(x) - T_m(x)\| < \epsilon'$, $\epsilon' > 0$ and $n, m > N$ implies $T_n(x)$ is Cauchy in Y. Since Y is complete, therefore there exists an element y such that the Cauchy sequence $T_n(x) \longrightarrow y \in Y$. Now, the limit y depends upon the choice of x because $\|T_n(x) - y\| \longrightarrow 0$. We can call this $y = T(x)$. Thus, we have $T_n(x) \longrightarrow T(x)$. To prove that $T(x) \in B(X, Y)$, we need to show that $T(x)$ is linear and bounded.

Linear:

$T(\alpha x + \beta y)$
$= \lim_{n\to\infty} T_n(\alpha x + \beta y)$
$= \lim_{n\to\infty} [\alpha T_n(x) + \beta T_n(y)]$
$= \lim_{n\to\infty} \alpha T_n(x) + \lim_{n\to\infty} \beta T_n(y)$
$= \alpha \lim_{n\to\infty} T_n(x) + \beta \lim_{n\to\infty} T_n(y)$
$= \alpha T(x) + \beta T(y)$
Bounded:
$\|T_n(x) - T(x)\|$
$= \left\| T_n(x) - \lim_{m\to\infty} T_m(x) \right\|$
$= \lim_{m\to\infty} \|T_n(x) - T_m(x)\|$
$\leq \lim_{m\to\infty} \|T_n - T_m\| \|x\|$
$= \|T_n(x) - T(x)\| \|x\|$
$< \epsilon \|x\|$

That is, $\|T_n(x) - T(x)\| \leq \epsilon \|x\|$. Hence the operator $(T_n - T)$ is bounded and $T_n - T \in B(X,Y)$. Since $T_n \in B(X,Y)$ and $B(X,Y)$ is closed under addition, therefore $T_n - (T_n - T) = T \in B(X,Y)$. ■

A similar property holds for compact X and $Y = \mathbb{R}$ or $\mathbb{C}$

Proposition 19 *If X is a compact metric space and Y is a complete metric space, then $C(X,Y)$, with the norm $\|f\| = \sup_x |f(x)|$, is complete.*

Proof. The existence of the norm is justified by the compactness criterion. Suppose (f_n) is a Cauchy sequence in $C(X,Y)$, so, as $n \longrightarrow \infty$, $\|f_n - f_m\| \longrightarrow 0$. In particular $(f_n(x))$ is a Cauchy sequence in Y for each $x \in X$ since $|f_n(x) - f_m(x)| = |(f_n - f_m)(x)| \leq \|f_n - f_m\| \|x\| \longrightarrow 0$ so it converges, say to $f(x) \in Y$. It remains to show that $f \in C(X,Y)$ and that $f_n \longrightarrow f$. We have that $|f_n(x) - f(x)| \leq \|f_n - f\| \|x\| \longrightarrow 0 \; \forall x$ i.e., $f_n \longrightarrow f$ uniformly. It remains only to show that f is continuous. For this, let $x_k \longrightarrow x$ in X and let $\epsilon > 0$. Pick N so that $\epsilon_N < \epsilon$. Since f_N is continuous, there exists $K \in \mathbb{N}$ such that $k \geq K$
$\implies |f_N(x_k) - f_N(x)| < \epsilon$. Hence $k \geq K$
$\implies |f(x_k) - f(x)|$
$= |f(x_k) - f_N(x) + f_N(x) - f(x) + f(x_k) - f(x_k)|$
$\leq |f(x_k) - f_N(x_k)| + |f_N(x_k) - f_N(x)| + |f(x) - f_N(x)| \longrightarrow 0$ ■

Similar results hold for functionals. Just like we can have for ourselves a norm space of bounded linear operators, we can have for ourselves a space of functionals. All we do is collect bounded functionals and have for ourselves a norm space.

Definition 20 *The collection of all functionals on a vector space V over $\mathbb{F}$ is called the **algebraic dual space** V^* of V.*

Since these are functionals, we can use our previous knowledge of functions to give us our addition and scalar multiplication binary operators. That is, for

$f_1 + f_2 \in V^*$, then

$$+ : V^* \times V^* \longrightarrow V^*$$

and

$$\cdot : \mathbb{F} \times V^* \longrightarrow V^*$$

Then

$$+ (f_1, f_2) = (f_1 + f_2)(x) = f_1(x) + f_2(x)$$

and

$$\cdot (\alpha, f) = (\alpha f)(x) = \alpha (f(x))$$

In this way, the additive identity is the zero function $\hat{O}(x) = 0$ to give us a vector space V^*.

We can go a step further ahead and consider the algebraic dual space $(V^*)^*$ of the dual space V^*, called the second algebraic dual V^{**}. This is the space of functionals on the dual space itself. We can move on and on but for now, second algebraic dual spaces will suffice.

Here is one purpose of considering the second algebraic dual space: we can define functionals of V^* as follows: $g(f)$. Remember, $g \in V^{**}$ and f acts as an input variable, much like $f \in V^*$ and acts on elements $x \in V$. Just like we can vary x to find different values for $f(x)$, likewise we can vary f to find different values of g. If we fix an $x \in V$, then one way of defining $g(f)$ is as follows: $g(f) = g_x(f) = f(x)$, with the subscript reminding us what to do with f.

This g is linear, keeping the x fixed.

Proof. $g(\alpha f_1 + \beta f_2)$
$= (\alpha f_1 + \beta f_2)(x)$
$= (\alpha f_1)(x) + (\beta f_2)(x)$
$= \alpha (f_1(x)) + \beta (f_2(x))$
$= \alpha g(f_1) + \beta g(f_2)$ ■

Since V^{**} is the collection of linear and bounded functionals on V^*, g_x really is an element of V^{**}. Just as we have kernels or null spaces of a specific mapping, that is, $\ker g = N(g) = \{x \mid g(x) = 0, x \in \mathcal{D}(g)\}$, we can also have a null space or a kernel of the entire vector space itself. In this case,

$$N(V) = \{x \mid f(x) = 0 \ , \forall f \in V^*\}$$

We can of course do the same with the algebraic dual space in which every such functional is considered. That is,

$$N(V^*) = \{x \mid g_x(f) = f(x) = 0 \ , \forall g \in V^{**}\} = N(V)$$

This is indeed a vector subspace of V.

Proof. Let $x, y \in N(V^*)$ such that $f(x) = f(y) = 0$ for any $f \in V^*$. Then, $f(\alpha x + \beta y) = 0$ hence $\alpha x + \beta y \in N(V^*)$ ■

In either case, $\dim N(g)$ and $\dim N(V^*) \leq n$ if $\dim V = n$. These facts follow from the fact that both are subspaces and from the fact that $\dim V = \dim V^*$

Notice the similarities and differences between the null space of an operator and the null space the vector space itself.

We now move to a justification of that subscripted x to consider a relationship between V and V^{**}. Let us define a mapping as follows and call it the **canonical mapping**: $C : V \longrightarrow V^{**}$ such that $C(x) = g_x$. This mapping is linear

Proof. $C(\alpha x + \beta y)$
$= g_{\alpha x+\beta y}$
$= f(\alpha x + \beta y)\ \forall f$
$= \alpha f(x) + \beta f(y)$
$= \alpha g_x + \beta g_y$
$= \alpha C(x) + \beta C(y)$ ■

In mathematical literature, this mapping is also called the canonical embedding of V into V^{**}. Since this operator C is linear and takes elements from a vector space to another vector space, we have for ourselves a vector space homomorphism! Provided that this mapping is bijective, we then have for ourselves an isomorphism. The choice of the word "embedding" should be clear from the choice of domain and range and the fact that we have a isomorphism to a subset of the codomain. This is also stated as follows: V is embeddable into V^{**}. The mapping C is one-to-one provided that the functionals f are injective. i.e. if we have two functionals g and h, then they are influenced because of different elements from the domain.

Proof. Let $C(x) = C(y)$. Then, $g_x = g_y$ and $f(x) = f(y)\ \forall f \implies x = y$ ■

Thus, if we limit the codomain to the range and assume that every functional f on V is injective, then we have for ourselves a bijective C and thus an isomorphism. If the codomain and the range are already the same, then we have for ourselves an isomorphism without limiting the codomain.

Definition 21 *Let $E = \{e_1, ..., e_n\}$ be a basis of X. $E^* = \{e_1^*, ..., e_n^*\}$ is an (algebraic)* ***dual basis*** *for the algebraic dual space X^* of X.*

Now this definition may not be exactly enlightening but was only mentioned to set some record straight. Let's look at it from a computational point of view (note that index i varies finitely). This is important because we want to be able to find the elements of a dual space. The computation follows the manner for operators. Thus, if $x = \sum_{i=1}^{n} \alpha_i e_i$, then $f(x) = \sum_{i=1}^{n} \alpha_i f(e_i) = \sum_{i=1}^{n} \alpha_i e_i^*$. For now, $f(e_i) = e_i^*$ is just a notation but we are trying to go in accord with the definition given above, as will hopefully be made clear. Notice that

$$\begin{bmatrix} e_1^* & e_2^* & ... & e_{n-1}^* & e_n^* \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \sum_{i=1}^{n} \alpha_i e_i^*$$

Clearly, our required $1 \times n$ matrix A is, therefore, $A = (e_{1i}^*) = (f(e_i))$. By the construction principle for linear maps (above), there exists a linear functional

$f_i = e_i^* \in E^*$ which maps e_i to 1 and the other basis vectors to 0. That is, for each basis e_k^*, $e_k^*(e_j) = f_k(e_j) = \delta_{kj}$ where δ_{ij} is the Kronecker delta function. Thus, if we have a vector $v = \sum_{i=1}^{n} \alpha_i e_i$ we must have

$$e_k^*(v) = f_k(v) = f_k\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i f_k(e_i) = \alpha_k$$

which shows that the linear functional e_i^* maps every vector of X to its i-th coordinate with respect to the basis B. In order to be able to thus say that for all $x \in X$, $x = \sum_{i=1}^{n} f_i(x) e_i$, we need to show that the E^* is a linearly independent set and this will be done in the theorem below but before that, notice how the space and its dual are connected and that this construction works for any basis $E = \{e_1, ..., e_n\}$.

Example 22 *The dual basis for the basis* $(1,0,0), (0,1,0)$ *and* $(0,0,1)$ *in* $\mathbb{R}^3$ *can be found as follows:* $(1,0,0)^T, (0,1,0)^T$ *and* $(0,0,1)^T$ *where the superscript* T *indicates the transpose of this vector. Justify it to yourself that these transposed vectors do indeed form functionals and the basis for the algebraic dual of* $\mathbb{R}^3$

Theorem 23 *Let* X *be a vector space and* $E = \{e_1, ..., e_n\}$ *be a basis of* X*. Then,* $E^* = \{e_1^*, ..., e_n^*\} = \{f_1, f_2, ..., f_n\}$ *is the basis for the agebraic dual* X^* *of* X *and* $\dim X = \dim X^* = n$

Lemma 24 *Let* X *be a finite dimensional vector space. If* $x_0 \in X$ *has the property that* $f(x_0) = 0$ *for all* $f \in X^*$*, then* $x_0 = 0$

Proof. Take $x_0 = \sum_{i=1}^{n} \alpha_i e_i$. For all $f \in X^*$, we have $0 = f(x_0)$

$= f\left(\sum_{i=1}^{n} \alpha_i e_i\right) = \sum_{i=1}^{n} \alpha_i f(e_i) = \sum_{i=1}^{n} \alpha_i e_i^* = 0$

Since $e_i^{*\prime}$'s are linearly independent, we must have $\alpha_i = 0 \forall i$. Hence $x_0 = \sum_{i=1}^{n} \alpha_i e_i \implies x_0 = 0$ ■

In addition to the algebraic dual space for vector spaces, we have an equivalent concept, called simply **dual space**, for norm spaces. Such a space will be denoted by X'

Corollary 25 *C is injective*

Proof. $C_x(f) = C_x(g) \implies f(x) = g(x) \implies (f-g)(x) = 0 \forall x \implies f - g = 0 \implies f = g$ ■

Theorem 26 (Hahn-Banach Theorem) *Let* $(X, \|.\|)$ *be a normed space and let* $Y \subseteq X$ *be a subspace. For any* $f \in X'$*, there exists* $\tilde{f} \in X^*$ *such that* $\tilde{f}$ *is an extension of* f *(*$\tilde{f}(y) = f(y)$ *for any* $y \in Y$*) and* $\left\|\tilde{f}\right\| = \|f\|$

Corollary 27 *Let X be a normed space and let $x_0 \neq 0$ be any element of X. Then, there exists a linear bounded functional $\tilde{f}$ on Xsuch that $\left\|\tilde{f}\right\| = 1$ and $\tilde{f}(x_0) = \|x_0\|$*

Proof. Consider the subspace Y consisting of $x = \alpha x_0$. Define f on Y by $f(x) = \alpha \|x_0\|$. f is bounded has norm $\|f\| = 1$ because $|f(x)| = |f(\alpha x_0)| = |\alpha| \|x_0\| = \|x\|$

From the Hahn-Banach theorem, $\|f\| = \left\|\tilde{f}\right\| = 1$. Further, $\tilde{f}(x_0) = f(x_0) = \|x_0\|$ ■

Corollary 28 *For every $x \in X$, $\|x\| = \sup\limits_{f \in X'} \frac{|f(x)|}{\|f\|}$*

Proof. $\sup\limits_{f \in X'} \frac{|f(x)|}{\|f\|} \geq \frac{\left|\tilde{f}(x)\right|}{\left\|\tilde{f}\right\|} = \frac{\|x\|}{1} = \|x\|$

Conversely, $|f(x)| \leq \|f\| \|x\|$ implies $\sup\limits_{f \in X'} \frac{|f(x)|}{\|f\|} \leq \|x\|$ ■

Corollary 29 *X"' and X are isometric. That is, $\|g_x\| = \|x\|$*

Proof. $\|g_x\| = \sup\limits_{f \in X'} \frac{|g_x(f)|}{\|f\|} = \sup\limits_{f \in X'} \frac{|f(x)|}{\|f\|} = \|x\|$ ■

Theorem 30 (Principle of uniform boundedness (Banach-Steinhaus)) *Let $(X, \|.\|_X)$ be a Banach space, $(Y, \|.\|_Y)$ a normed space and $T_n : X \longrightarrow Y$ a bounded operator for each $n \in N$. Suppose that for any $x \in X$ there exists $C_x > 0$ such that $\|T_n x\|_Y \leq C_x$ for all n. Then there exists $C > 0$ such that $\|T_n\| \leq C$ for all n.*

Theorem 31 *If (x_n) is a sequence in a Banach space and $(f(x_n))$ is bounded for all $f \in X'$, show that $\|x_n\|$ is bounded.*

Proof. We will apply the uniform boundedness principle to the dual space X^*. This is complete, whether or not X is. The role of T_n will be played by $\hat{x}_n \in X^{**}$. Recall that $\hat{x}_n$ is defined as the bounded linear functional on X^* for which $\hat{x}_n(f) = f(x_n)$ $(f \in X^*)$. The assumption that $(f(x_n))$ is bounded means that for any vector f in our space X^* the sequence $\hat{x}_n(f)$ is bounded. Using the uniform boundedness principle we get that there exists C such that $\|\hat{x}_n\| \leq C$ for all n. From the corollary of Hahn-Banach theorem, $\|x_n\| = \|\hat{x}_n\|$. ■

2 Algebras

For a vector space V over $\mathbb{F}$ equipped with an additional binary operation from $\cdot : V \times V \longrightarrow A$ is an **algebra** over $\mathbb{K}$ if the following identities hold for any three elements x, y, and z of V, and all scalars α of $\mathbb{K}$

1. $(x \cdot y \cdot) z = x \cdot (y \cdot z)$ (Associativity)

2. $(x + y) \cdot z = x \cdot z + y \cdot z$ (Right Distributivity)

3. $x \cdot (y + z) = x \cdot \ y + x \cdot z$ (Left Distributivity)

4. $(\alpha x \cdot y) = \alpha(x \cdot y) = \alpha(x \cdot \alpha y)$ (Compatibility with scalars)

These three axioms are another way of saying that the binary operation is bilinear. An algebra over $\mathbb{K}$ is sometimes also called a $\mathbb{K}$-algebra, and $\mathbb{K}$ is called the base field of V. The binary operation is often referred to as multiplication in V. For example, for the vector space $C[a, b]$ of continuous functions, the ordinary multiplication of functions satisfies the above. If this multiplication is associative and if there are inverses such that the for identity $e = xx^{-1}$, then V is an **associative division algebra.**

We can turn a norm space into an algebra if

$$\|xy\| \leq \|x\| \|y\|$$

holds for all x, y

If the space is complete, then the norm algebra is referred to as a Banach Algebra.

Example 32 *Let H be a Hilbert space. The norm operator on the algebra $B(H)$ of bounded linear operators on H is a norm space with multiplication defined by composition. Then, $\|TS\| = \sup\limits_{\|x\|=1} \|T(S(x))\| \leq \sup\limits_{\|x\|=1} \|T\| \|S\| = \|T\| \|S\|$. Since $B(H, H) = B(H)$, therefore $B(H)$ is complete and forms a Banach Algebra*

Example 33 *Corresponding results for $C(X)$, continuous functionals defined on a compact space X, with the pointwise multiplication $(fg)(x) = f(x)g(x)$, is a Banach algebra. First, collection of bounded functionals forms a vector space (dual space). Axioms for algebra are routine to verify. Norm can be defined as $\|f\| = \sup\limits_{\|x\|=1} |f(x)|$. The unit element is the function $e(x) \equiv 1$. This has norm 1. Finally, $\|fg\| = \sup\limits_{\|x\|=1} |(fg)(x)|$*

$= \sup\limits_{\|x\|=1} |f(x) g(x)| \leq \sup\limits_{\|x\|=1} |f(x)| \sup\limits_{\|x\|=1} |g(x)| = \|f\| \|g\|$. This space is complete since X compact and $Y = \mathbb{R}$ or $\mathbb{C}$ is complete

Example 34 *If A is a normed (resp. Banach) algebra, $A^n := A \oplus ... \oplus A$ (n copies of A) with the norm defined by $\|(x_1, .., x_n)\| = \max\limits_{1 \leq i \leq n} \{\|x_i\|\}$ is a normed (resp. Banach) algebra.*

Proposition 35 *Every isometric function between two Banach Algebras is injective hence*

Proof. $\|x\| = \|T(x)\|$, then $T(x) = 0$ implies $\|T(x)\| = \|x\| = 0$ implies $x = 0$ ■

Proposition 36 *Every finite dimensional algebra is complete*

Proof. Every finite dimensional norm space is complete. ■

Proposition 37 $\|e\| \geq 1$

Proof. $\|x\| = \|xe\| \leq \|x\| \|e\|$
$\|e\| \geq 1$ ■

Proposition 38 $\|x\|^{-1} \leq \|x^{-1}\|$

Proof. $\|xx^{-1}\| = \|e\| \leq \|x\| \|x^{-1}\|$
That is, $1 \leq \|x\| \|x^{-1}\|$
$\|x\|^{-1} \leq \|x\|^{-1} \|x\| \|x^{-1}\|$
$\|x\|^{-1} \leq \|x^{-1}\|$ ■

For Banach algebras, the condition of being non-unital is superfluous and the unitization process can be applied to Banach algebras that are already unital, too. Let A be a Banach algebra. Set $A_1 := A \times \mathbb{C}$ and define the ordinary operations by $(x, \lambda)(y, \mu) := (xy + \lambda y + \mu x, \lambda\mu)$ and $|(x, \lambda)| := \|x\| + |\lambda|$ for all $x, y \in A$ and $\lambda, \mu \in \mathbb{C}$. The algebra A_1 is called the **Banach algebra unitization** of A. It is straight-forward to check that the above definitions give us an algebra with unit $(0, 1)$. The inequality is also easily seen to be valid, using which we can show that any Cauchy sequence (x_n, λ_n) converges to (x, λ) if $x_n \longrightarrow x$ and $\lambda_n \longrightarrow \lambda$. We adopt another approach using the familiar completion process.

Proof. First, we focus on the construction of $(A_1, \|.\|_1)$. The idea is to get $\alpha_n = (x_n, \lambda_n)$ and $\beta_n = (y_n, \upsilon_n)$ Cauchy sequences and call them equivalent if $\|\alpha_n - \beta_n\|_1 \longrightarrow 0$. Let x_n and y_n be Cauchy sequences in A. We will call two Cauchy sequences equivalent if they have the same limit i.e.

$$\lim_{n\to\infty} \|x_n - y_n\| = 0$$

This will be written as $(x_n) \sim (y_n)$. We can then gather all such equivalent sequences and form an equivalent class. Indeed, $(x_n) \sim (x_n)$ is trivial, so this relation is reflexive. Also, since the arguments of a norm function are symmetric, the relation $\sim$ is symmetric. Finally, if $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$, we have

$$\|x_n - z_n\| \leq \|x_n - y_n\| + \|x_n, -y_n\|$$

Taking limits on both sides and using the fact that the norm function is always positive, we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0$$

so that $(x_n) \sim (z_n)$, implying transitivity. Now, the set C is already complete and, therefore, already contains well-defined equivalent Cauchy seqeunces.

Combining these with our previous equivalence class, we can have another equivalence class, the construction of which is similar basing it on component-wise addition and scalar multiplication with the above vector multiplication. Thus, we can have for ourselves an equivalence class $\left(\hat{x}, \hat{\lambda}\right) = \left\{\left(\bar{x}_n, \bar{\lambda}_n\right)\right\}$ of Cauchy sequences. We can collect all such equivalence classes $\hat{x}, \hat{y}, ...$ and form the set A_1. For this set, we can have the norm

$$\|\hat{x} - \hat{y}\|_1 := \lim_{n\to\infty} \|x_n - y_n\| + \lim_{n\to\infty} |\lambda_n - \upsilon_n|$$

where $(x_n, \lambda_n) \in \hat{x}$ and $(y_n, \upsilon_n) \in \hat{y}$. Note that this is not equal to zero since x_n and y_n are members of a different equivalence class. Furthermore, it is trivial to show that this newly defined norm satisfies the axioms for a norm.

To show that this limit is well-defined or that this definition is sensible and not ambiguous with different results for the same choice of inputs, we will first show that this limit exists and then show that it is independent of the choice of representatives. First, we have

$$\|x_n - y_n\|+|\lambda_n - \upsilon_n| \leq \|x_n - x_m\|+|\lambda_n - \lambda_m|+\|x_m - y_m\|+|\lambda_m - \upsilon_m|+\|y_m - y_n\|+|\upsilon_m - \upsilon_n|$$

$$\implies$$

$$\|x_n - y_n\|+|\lambda_n - \upsilon_n|-\|x_m - y_m\|-|\lambda_m - \upsilon_m| \leq \|x_n - x_m\|+|\lambda_n - \lambda_m|+\|y_m - y_n\|+|\upsilon_m - \upsilon_n|$$

Similarly,

$$\|x_m - y_m\|+|\lambda_m - \upsilon_m| \leq \|x_m - x_n\|+|\lambda_n - \lambda_m|+\|x_n - y_n\|+|\lambda_n - \upsilon_n|+\|y_n - y_m\|+|\upsilon_m - \upsilon_n|$$

$$\implies$$

$$\|x_m - y_m\|+|\lambda_m - \upsilon_m|-\|x_n - y_n\|-|\lambda_n - \upsilon_n| \leq \|x_n - x_m\|+|\lambda_n - \lambda_m|+\|y_m - y_n\|+|\upsilon_m - \upsilon_n|$$

$$\implies$$

$$-\left(\|x_n - x_m\| + |\lambda_n - \lambda_m| + \|y_n - y_m\| + |\upsilon_m - \upsilon_n|\right) \geq \|x_n - y_n\|+|\lambda_n - \upsilon_n|-\|x_m - y_m\|-|\lambda_m - \upsilon_m|$$

this is basically $b \geq a$ and $-b \geq a$ so that we have $|a| \leq b$. Hence,

$$|\|x_n - y_n\| + |\lambda_n - \upsilon_n| - \|x_m - y_m\| - |\lambda_m - \upsilon_m|| \leq \|x_n - x_m\|+|\lambda_n - \lambda_m|+\|y_n - y_m\|+|\upsilon_m - \upsilon_n|$$

Now, since x_n is Cauchy, we have $\|x_n - x_m\| < \epsilon/4$ and similarly $\|y_n - y_m\| < \epsilon/4$, $|\upsilon_m - \upsilon_n| < \epsilon/4$ and $|\lambda_n - \lambda_m| < \epsilon/4$. This in turn implies that for $n, m > N$

$$|\|x_n - y_n\| + |\lambda_n - \upsilon_n| - \|x_m - y_m\| - |\lambda_m - \upsilon_m|| < \epsilon$$

so that

$$\lim_{n\to\infty} \|x_n - y_n\| + \lim_{n\to\infty} |\lambda_n - \upsilon_n| = \lim_{m\to\infty} \|x_m - y_m\| + \lim_{n\to\infty} |\lambda_m - \upsilon_m|$$

Hence, $\|\hat{x} - \hat{y}\|_1$ is just as valid for any Cauchy sequence. It is now routine to show that $(A_1, \|.\|_1)$ is a Banach space.

We have just proved that for any Banach space $(A, \|.\|)$, we will have another metric space $(A_1, \|.\|_1)$ by accounting for the limits of the Cauchy sequences, made possible by clumping all Cauchy sequences with common limits in both arguements. Let $T : A \longrightarrow A_1$ be a mapping such that $T(a) = (\hat{a}, 0)$ where $\hat{a}$ is an equivalence class of Cauchy sequences. This is an isometry. ■

Proposition 39 *Multiplication is continuous in Banach algebras*

Proof. Let $x_n \longrightarrow x$ and $y_n \longrightarrow y$.

$\|x_n y_n - xy\| \leq \|x_n - x\| \|y_n\| + \|y_n - y\| \|x_n\| \longrightarrow 0$ ■

Given a Banach algebra $(A, \|.\|)$, for every real number $r \geq 1$, $(A, r\|.\|)$ is a Banach algebra too. Thus the norm of the unit is not necessarily 1 in unital Banach algebras. However, the norm of an arbitrary Banach algebra can be replaced by another norm so that the new norm of the unit to be 1.

Proposition 40 *Let $(A, \|.\|)$ be a unital Banach algebra. Then there exists a norm $\|.\|_o$ on A such that*

(i) The norms $\|.\|$ and $\|.\|_o$ are equivalent on A,

(ii) $(A, \|.\|_o)$ is a Banach algebra,

(iii) $\|e\|_o = 1$.

Proof. We have already proved that $B(A)$ is an algebra. We embed A into $B(A)$ by left multiplication; Let $L_x(y) = xy$. Then, $L_x(y_1 + y_2) = L_x(y_1) + L_x(y_2)$ and $L_x(\alpha y) = \alpha L_x(y)$ so that this operator is continuous. Hence $B(A) \neq \varnothing$

Next, let $L : A \longrightarrow B(A)$ be defined as $L(x) = L_x$

Then, $L(x + y) = L_{x+y}$

Now, $L_{x+y}(z) = (x + y) z = xz + xz = L_x(z) + L_y(z)$ for any z so that $L(x + y) = L(x) + L(y)$

Similarly, $L(\alpha y) = \alpha L(y)$

Hence L is a homomorphism. We define the norm $\|.\|_o$ on A to be the restriction of the operator norm of $B(A)$ to the image of A, that is $\|x\|_o := \|L_x\| = \sup\limits_{\|y\| \leq 1} \|xy\|$

For $\|y\| \leq 1$, we have $\|xy\| \leq \|x\| \|y\| \leq \|x\|$. This shows that $\|x\|_o \leq \|x\|$. On the other hand, we have

$$\frac{\|x\|}{\|e\|} = \frac{\|xe\|}{\|e\|} \leq \sup_{y \neq 0} \|xy\| = \|x\|_o$$

This shows that $\|x\| \leq \|e\| \|x\|_o$ for all $x \in A$ and completes the proof of (i). It follows from (i) that A is a closed subalgebra of $B(A)$, so it is a Banach algebra with the new norm $\|.\|_o$. Part (iii) is clear from the definition. ■

$x \in A$ is called **invertible** if there exists $y \in A$ so that $xy = yx = e$. This y is unique

Proof. $y_1 = y_1 e = y_1 x y_2 = y_2$ ■

The set of invertible elements is denoted by $G(A)$. This is a group

Proof. If $x, y \in G(A)$, then $x^{-1}, y^{-1} \in A$ and $(xy)\left(y^{-1}x^{-1}\right) = e$

$\implies xy \in G(A)$

Associative carries over

$ee = e \implies e^{-1} = e \implies e \in G(A)$

$x \in G(A) \implies x^{-1} \in A \implies xx^{-1} = x^{-1}x = e \implies x^{-1} \in G(A)$ since $\left(x^{-1}\right)^{-1}$ is x ∎

Proposition 41 *Let A be a Banach algebra. If $x \in A$, $\|x\| < 1$, then $e - x \in G(A)$ and $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n$*

Proof. From $\|x^n\| \leq \|x\|^n$ and $\|x\| < 1$, we have that $\sum_{n=0}^{\infty} \|x^n\|$ converges as a geometric series with ratio less than 1. An absolutely convergent series is convergent in a Banach space. Thus, $\sum_{n=0}^{\infty} x^n$ exists. On the other hand, $(e-x)\sum_{n=0}^{\infty} x^n = (e-x)\lim_{k\to\infty}\sum_{n=0}^{k} x^n$

Since multiplication is continuous, this equals $\lim_{k\to\infty}(e-x)\sum_{n=0}^{k} x^n$

$= \lim_{k\to\infty}\left(e\sum_{n=0}^{k} x^n - x\sum_{n=0}^{k} x^n\right)$

$= \lim_{k\to\infty}\left(\sum_{n=0}^{k} x^n - \sum_{n=0}^{k} x^{n+1}\right)$

$= \lim_{k\to\infty}\left(e - x^{k+1}\right) = e$

Similarly, $\sum_{n=0}^{\infty} x^n(e-x) = e$

Hence $(e-x)^{-1} = \sum_{n=0}^{\infty} x^n$ ∎

Corollary 42 $\|x - e\| < 1$ *implies* $x^{-1} = \sum_{n=0}^{\infty} (e-x)^n$

Proof. Take $y = e - x$

Then, $(e-y)^{-1} = x^{-1} = \sum_{n=0}^{\infty} y^n = e + \sum_{n=1}^{\infty} (e-x)^n$ ∎

Proposition 43 *Inversion is a continuous process*

Proof. Let a be an invertible element. Let $\|x-a\| < 1/\left\|a^{-1}\right\|$. Then, $\left\|xa^{-1}-e\right\| = \left\|(x-a)\,a^{-1}\right\| \leq \|x-a\|\left\|a^{-1}\right\| < 1$ hence by previous corollary applied on xa^{-1} instead of x, we get $ax^{-1} = \sum_{n=0}^{\infty}\left(e-xa^{-1}\right)^{n}$ or $x^{-1} = \sum_{n=0}^{\infty}a^{-1}\left(e-xa^{-1}\right)^{n}$.

Hence $\left\|x^{-1}-a^{-1}\right\| = \left\|\sum_{n=1}^{\infty}a^{-1}\left(e-xa^{-1}\right)^{n}\right\|$

$\leq \left\|a^{-1}\right\|\left\|\sum_{n=1}^{\infty}\left(e-xa^{-1}\right)^{n}\right\|$

$\leq \left\|a^{-1}\right\|\sum_{n=1}^{\infty}\left\|a^{-1}\right\|\|a-x\|\left\|e-xa^{-1}\right\|^{n-1}$

$= \left\|a^{-1}\right\|^{2}\|a-x\|\sum_{n=1}^{\infty}\left\|e-xa^{-1}\right\|^{n-1}$

The latter is a geometric series with ratio $\left\|e-xa^{-1}\right\|$ and is convergent since $1 > \|x-a\|\left\|a^{-1}\right\| \geq \left\|(x-a)\,a^{-1}\right\| = \left\|xa^{-1}-e\right\|$. Using this, we can have for ourselves the required ϵ. Thus, for any $\epsilon > 0$, we can have $\delta = 1/\left\|a^{-1}\right\|$, implying continuity. ■

Proposition 44 *$G(A)$ is open*

Proof. Let $a \in G$. Let $\|x-a\| < 1/\left\|a^{-1}\right\|$.

Then, $\left\|xa^{-1}-e\right\| = \left\|(x-a)\,a^{-1}\right\| \leq \|x-a\|\left\|a^{-1}\right\| < 1$ hence $xa^{-1}-e+e = xa^{-1} \in G(A) \implies x \in G(A)$. ■

Exercise 45 *Construct a sequence of invertible elements in $C[-1,1]$ which converge to a non-invertible element in $C[-1,1]$*

Solution 46 *Consider functions $f_n = \frac{1}{n}e$ where e is unit. Then, $f_n \longrightarrow 0$ but this limit not invertible.*

Lemma 47 *If $x_n \in G(A)$ such that $x_n \longrightarrow x \in \delta G(A) = \overline{G(A)} \cap \overline{G(A)^{c}}$, then $\left\|x_n^{-1}\right\| \longrightarrow \infty$*

Proof. Assume the contrary that $\left\|x_n^{-1}\right\| \leq k$ for every n. Then, from $\|x_n - x\| < 1/k$, we have $\left\|e - x_n^{-1}x\right\| \leq \left\|x_n^{-1}\right\|\|x_n - x\| < 1$ so that $x_n^{-1}x$ is invertible
$\implies x \in G(A)$ implying that $G(A)$ is closed. ■

Exercise 48 *Let A be an algebra, with unit e. Is the following true or false?*

1. *$fx = x$ for all $x \in A \implies f = e$;*
2. *$0x = 0$ for all $x \in A$; true*
3. *$xy = 0 \implies x = 0$ or $y = 0$; false*
4. *$xy = zx = e \implies x \in G(A)$ and $y = z = x^{-1}$ True*
5. *$xy; yx \in G(A) \implies x; y \in G(A)$;*
6. *$xy = e \implies x \in G(A)$ or $y \in G(A)$;*

3 Analytic Maps

Definition 49 *A mapping $g : \Omega \subseteq \mathbb{C} \longrightarrow A$ is **analytic** at $\lambda_0 \in \Omega$ if*

$$\lim_{\lambda \to \lambda_0} \frac{g(\lambda) - g(\lambda_0)}{\lambda - \lambda_0}$$

exists. g is said to be analytic on Ω if it is analytic at every point of Ω

Proposition 50 *$g : \rho(a) \longrightarrow A$ such that $g(\lambda) = (a - \lambda e)^{-1}$ is analytic*

Proof. $\frac{g(\lambda)-g(\lambda_0)}{\lambda-\lambda_0} = \frac{(a-\lambda e)^{-1}-(a-\lambda_0 e)^{-1}}{\lambda-\lambda_0} = \frac{(a-\lambda e)^{-1}\left(e-(a-\lambda e)(a-\lambda_0 e)^{-1}\right)}{\lambda-\lambda_0}$
$= \frac{(a-\lambda e)^{-1}((a-\lambda_0)-(a-\lambda e))(a-\lambda_0 e)^{-1}}{\lambda-\lambda_0}$
$= \frac{(a-\lambda e)^{-1}(a-\lambda_0 e-a-\lambda e)(a-\lambda_0 e)^{-1}}{\lambda-\lambda_0}$
$= \frac{(a-\lambda e)^{-1}(\lambda_0 e-\lambda e)(a-\lambda_0 e)^{-1}}{\lambda-\lambda_0}$
$= (a - \lambda e)^{-1} (a - \lambda_0 e)^{-1}$

Now, $\|(a - \lambda e) - (a - \lambda_0 e)\| \leq |\lambda - \lambda_0| < \delta$ says that $a - \lambda e \longrightarrow a - \lambda_0 e$ if $\lambda \longrightarrow \lambda_0$

Also, inverse function is continuous and, therefore, $(a - \lambda e)^{-1}$ is continuous.

Hence $\lim\limits_{\lambda \to \lambda_0} \frac{g(\lambda)-g(\lambda_0)}{\lambda-\lambda_0} = (a - \lambda_0 e)^{-2}$ ■

Theorem 51 (Liouville's Theorem) *Suppose $g : \mathbb{C} \longrightarrow A$ is analytic and bounded. Then, g is constant.*

Proof. Let $\phi \in A'$ be a linear bounded functional on A. Define $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(\lambda) = \phi(g(\lambda))$ for $\lambda \in \mathbb{C}$. Since g is analytic, then for $\lambda_0 \in \mathbb{C}$, $\lim\limits_{\lambda \to \lambda_0} \frac{g(\lambda)-g(\lambda_0)}{\lambda-\lambda_0} = g'(\lambda_0)$ exists. Now, $\frac{f(\lambda)-f(\lambda_0)}{\lambda-\lambda_0} = \frac{\phi(g(\lambda))-\phi(g(\lambda_0))}{\lambda-\lambda_0} = \phi\left(\frac{g(\lambda)-g(\lambda_0)}{\lambda-\lambda_0}\right)$ since ϕ is linear. Then,

$\lim\limits_{\lambda \to \lambda_0} \frac{f(\lambda)-f(\lambda_0)}{\lambda-\lambda_0} = \lim\limits_{\lambda \to \lambda_0} \phi\left(\frac{g(\lambda)-g(\lambda_0)}{\lambda-\lambda_0}\right) = \phi(g'(\lambda_0))$. Hence f is analytic. Further, f is bounded because
$|f(\lambda)| = |\phi(g(\lambda))|$
$\leq \|\phi\| \|g(\lambda)\|$
$\leq \|\phi\| M$ since g is bounded

Thus, by classical Louiville's theorem, f is constant. For $\lambda \neq \lambda_0$, $\phi(g(\lambda) - g(\lambda_0)) = \phi(g(\lambda)) - \phi(g(\lambda_0)) = f(\lambda) - f(\lambda_0) = 0$ since f is constant. That is, $\phi(g(\lambda) - g(\lambda_0)) = 0$ for all ϕ. Hence $g(\lambda) = g(\lambda_0)$ ■

4 Spectrum

Let $x \in A$. Then we define $\rho(x) = \{\lambda : x - \lambda e \in G(A)\}$, $\sigma(x) = \{\lambda : x - \lambda e \notin G(A)\}$, $r(x) = \sup\limits_{\lambda \in \sigma(x)} |\lambda|$

Theorem 52 *$\sigma(x)$ is closed and bounded*

Proof. We prove that $\rho(x) = \mathbb{C} - \sigma(x)$ is open. Let $\lambda_0 \in \rho(x)$. Then, $x - \lambda_0 e$ is invertible. There is a neighbourhood $N \subset A$ consisting wholly of invertible elements. Now, for a fixed x, the mapping $\lambda \longmapsto x - \lambda e$ is continuous. Hence all $x - \lambda e$ with λ close to λ_0, say $|\lambda - \lambda_0| < \delta$ lie in N so that these $x - \lambda e$ are invertible. This means that the corresponding λ belong to $\rho(x)$. Thus, every point of $\rho(x)$ is an interioir point, implying that $\sigma(x)$ is closed.

Next, if $|\lambda| > \|x\|$, then $\left\|\lambda^{-1}x\right\| < 1$ so that $e - \lambda^{-1}x$ is invertible. Hence $\lambda\left(e - \lambda^{-1}x\right) = \lambda e - x \in G(A) \implies \lambda \in \rho(x)$.

That is, $\lambda \in \sigma(x) \implies |\lambda| \leq \|x\|$ ■

Corollary 53 $r(x) \leq \|x\|$

Theorem 54 $\sigma(x) \neq \varnothing$

Proof. Let $\sigma(x) = \varnothing$ and $|\lambda| > \|x\|$. Then, $(x - \lambda e)^{-1} = -\sum \lambda^{-n-1}x^n$. For any non-zero $f \in A'$, define $g : \rho(x) \longrightarrow \mathbb{C}$ such that $g(z) = f\left((x - \lambda e)^{-1}\right)$. This function is defined on all $\mathbb{C}$.

Then, $f\left((x - \lambda e)^{-1}\right) = f\left(-\sum \lambda^{-n-1}x^n\right) = -\sum \lambda^{-n-1} f(x^n)$ hence $g(z)$ has a power series representation about every point and is thus holomorphic. If $|\lambda| \geq 2\|x\|$, then $|g(z)| = \left|f\left(-\sum \lambda^{-n-1}x^n\right)\right| \leq \|f\| \sum |\lambda|^{-n-1} \|x\|^n \leq \frac{\|f\|}{|\lambda|}\sum 2^{-n} = 2\frac{\|f\|}{|\lambda|}$ so g is bounded. By Louivelle's theorem, g is constant. Since $g(z) \longrightarrow 0$ as $z \longrightarrow \infty$, $g(z) = 0$. Thus, $f(y) = 0$ for all f implies $y = (x - \lambda e)^{-1} = 0$, a contradiction ■

Proposition 55 $\lambda \in \sigma(x) \implies \lambda^n \in \sigma(x^n)$

Proof. We prove the contrapositive. Let $n \in \mathbb{N}$ and let $\lambda \in \mathbb{C}$ be such that $\lambda^n \in \rho(x^n)$. We can write $x^n - \lambda^n e = (x - \lambda e)\left(\lambda^{n-1}e + \lambda^{n-2}x + ... + x^{n-1}\right)$ and now multiplication from the right by $(x^n - \lambda^n e)^{-1}$ shows that $x - \lambda e$ has a right inverse. A similar calculation provides a left inverse also, so it follows that $\lambda \in \rho(x)$ ■

Proposition 56 *For* $\Delta = \{\lambda : |\lambda| < 1/r(x)\}$, *then* $e - \lambda x \in G(A)$

Proof. It is inherently assumed that $r(x) \neq 0$.

We can combine $r(x) \leq \|x\|$ and $0 \neq \lambda \in \Delta$, $|\lambda| < 1/r(x)$ or $1/|\lambda| > r(x)$ to get $r(x) \leq \|x\| < 1/|\lambda|$

Focusing on the right side, we have $\|x\lambda\| < 1$. The result follows. ■

Theorem 57 $r(x) = \lim \|x^n\|^{1/n}$

Proof. We prove $\limsup \|x^n\|^{1/n} \leq r(x) \leq \liminf \|x^n\|^{1/n}$

$\lambda \in \sigma(x) \implies \lambda^n \in \sigma(x^n)$ so that $|\lambda^n| = |\lambda|^n \leq \|x^n\|$ for all $n \in \mathbb{N}$. Hence $r(x) \leq \inf_n \|x^n\|^n$, for all $n \in \mathbb{N}$, which implies the right hand side inequality

Let $\phi \in A'$ be any functional. From above, we have that $\lambda \in \Delta = \{\lambda : |\lambda| < 1/r(x)\}$, then $e - \lambda x \in G(A)$. Hence can define the function $f : \Delta \longrightarrow \mathbb{C}$ such that $f(\lambda) = \phi\left((e-\lambda x)^{-1}\right) = \phi\left(\sum \lambda^n x^n\right) = \sum \lambda^n \phi(x^n)$ so that f is analytic. Moreover, the sequence $\lambda^n \phi(x^n) \longrightarrow 0$ for each $\lambda \in \Delta$ and is bounded. Thus, by the principle of uniform boundedness, $\lambda^n x^n$ is bounded. Hence $\|\lambda^n x^n\| \leq M$ for all n and $\|x^n\|^{1/n} \leq M^{1/n}/|\lambda|$ and consequently $\limsup \|x^n\|^{1/n} \leq 1/|\lambda|$

In summary if $r(x) < 1/|\lambda|$, then $\limsup \|x^n\|^{1/n} \leq 1/|\lambda|$. It follows that $\limsup \|x^n\|^{1/n} \leq r(x)$ ■

Proposition 58 $\sigma(ab) = \sigma(ba)$

Proof. We prove that $e - ba$ is invertible if and only if $e - ab$ is. The rest follows. If $(e-ab)c = e$, then $(e+bca)(e-ba) = e - ba + bca - bcaba$. Given that $c - cab = e$, then $e - ba + bca - bcaba = e - ba + b(c - cab)a = e$ and similarly the converse ■

Corollary 59 $r(ab) = r(ba)$

Theorem 60 (Spectral Mapping Theorem) *For a polynomial p on $\mathbb{C}$, define $p(\sigma(x))$ as $\{p(z) : z \in \sigma(A)\}$. Then $p(\sigma(x)) = \sigma(p(x))$*

Proof. Let $z, \alpha \in \mathbb{C}$. Compare the factorisations

$$p(z) - \alpha = c\prod_{i=1}^{n}(z - \beta_i(\alpha))$$

$$p(x) - \alpha e = e\prod_{i=1}^{n}(x - \beta_i(\alpha)e)$$

Here the coefficients c and $\beta_i(\alpha)$ are determined by p and α. When $\alpha \in \rho(p(x))$ then $p(x) - \alpha e$ is invertible, which implies that all $x - \beta_i(\alpha)e$ must be invertible. Hence $\alpha \in \sigma(p(x))$ implies that at least one of the $x - \beta_i(\alpha)e$ is not invertible, so that $\beta_i(\alpha) \in \sigma(x)$ for at least one i. Hence $p(\beta_i(\alpha)) - \alpha = 0$, i.e., $\alpha \in p(\sigma(x))$. This proves the inclusion $p(\sigma(x)) \supseteq \sigma(p(x))$

Conversely, when $\alpha \in p(\sigma(x))$ then $\alpha = p(z)$ for some $z \in \sigma(x)$, so that for some i one must have $\beta_i(\alpha) = z$ for this particular z. Hence$\beta_i(\alpha) \in \sigma(x)$, so that $x - \beta_i(\alpha)$ is not invertible, implying that $p(x) - \alpha e$ is not invertible, so that $\in \sigma(p(x))$. This shows that $p(\sigma(x)) \subseteq \sigma(p(x))$ ■

Corollary 61 $\sigma(x^n) = \sigma(x)^n$

Proof. For $p(x) = x^n$ and $p(\sigma(x)) = \sigma(p(x))$, we have that $\sigma(x^n) = p(\sigma(x)) = \sigma(x)^n$ ■

Proposition 62 *The linear combination of injective operators is injective*

Proof. It is straightforward to show that a linear combination of linear operators is linear. Let $T,\ S$ be bijective on the same domain. Then, $T(y)=0$ and $S(y)=0$ implies $y=0$ and we need $(\alpha T+\beta S)(x)=0$ implies $x=0$. If $x\neq y$, then $T(x)=0$ so that $\ker T\neq\{0\}$ ■

Exercise 63 *Let $T:l^2\longrightarrow l^2$ such that $T(x_1,x_2,...)=\left(x_1,\frac{x_2}{2},\frac{x_3}{3},...\right)$. What is $\sigma(T)$ and $r(T)$?*

Solution 64 *A bit of trickery: T is injective, λI is injective $\forall\lambda\neq0$. Hence $T-\lambda I$ is not invertible for $\lambda=0$. But this is not the only value since we are not accounting for surjectivity.*

If $\lambda\in\sigma(T)$, then $(T-\lambda I)$ is not invertible. That is, $\ker(T-\lambda I)\neq\{0\}$ $\Longrightarrow (T-\lambda I)(x)=\left(x_1-\lambda x_1,\frac{x_2}{2}-\lambda x_2,\frac{x_3}{3}-\lambda x_3,...\right)=(0,0,...)$

$\Longrightarrow \lambda=\frac{1}{n}$ *if $x_i\neq0$ for all i*

Hence $\left\{\frac{1}{n}:n\in\mathbb{N}\right\}=\sigma(T)$. On the other hand, $1\leq\limsup\sqrt[2]{\Sigma\left|\frac{x_n}{n^k}\right|^2}\leq r(T)\leq\liminf\sqrt[2]{\Sigma\left|\frac{x_n}{n^k}\right|^2}\leq\lim\zeta(2k)=1$ *so that $r(T)=1$ where ζ is the Riemann Zeta function.*

Theorem 65 (Gelfand-Mazur theorem) *If $G(A)=A\backslash\{0\}$, then $A\cong\mathbb{C}$*

Proof. We can pick a number $\lambda\in\sigma(x)$ for each $x\in A$. So $x-\lambda e\in G(A)$ but the only non-invertible element of A is the zero vector, so we must have that $x=\lambda e$. The map $T:A\longrightarrow\mathbb{C}$ such that $x=\lambda e\longmapsto\lambda$ has the desired properties. That is, $f(\lambda e)=\lambda$.

This map is well defined and injective since $\lambda_1e=\lambda_2e$ if and only if $\lambda_1=\lambda_2$. Next, for every λ, we can have $\lambda e=x$ so that f is onto. Linearity is clear and $\|x\|=|\lambda|$ ■

Theorem 66 *Let A be a unital Banach Algebra. If there exists $k<\infty$ such that $\|x\|\|y\|\leq k\|xy\|$, then A is isometrically isomorphic to $\mathbb{C}$*

Proof. Let $x_n\in G(A)$ such that $x_n\longrightarrow x\in G(A)^d$, then $\left\|x_n^{-1}\right\|\longrightarrow\infty$ so that $\|x_n\|\left\|x_n^{-1}\right\|\leq k\left\|x_n^{-1}x_n\right\|=k$

$\Longrightarrow \|x_n\|\leq k/\left\|x_n^{-1}\right\|\longrightarrow0$

Hence any boundary point of $G(A)$ is zero and not invertible. Now, for arbitrary $x\in A$ and $\lambda\in\overline{\sigma(x)}\cap\overline{\rho(x)}\subseteq\mathbb{C}$, $\lambda e-x\in\overline{G(A)}\cap\overline{G(A)^c}$ so that $\lambda e-x=0$

$\Longrightarrow A=\{\lambda e:\lambda\in\mathbb{C}\}$

Thus we can define $f:A\longrightarrow\mathbb{C}$ such that $x=\lambda e\longmapsto\lambda$ ■

5 Multiplicative linear functionals

Definition 67 *A linear functional $f:A\longrightarrow\mathbb{C}$ is **mulitplicative** if $f(xy)=f(x)f(y)$ for all $x,y\in A$*

Example 68 *For* $A = \mathbb{C}$, $f(z) = z$

Example 69 *For* $A = \mathbb{C}^n$, $f_i(z_1, z_2, ..., z_n) = z_i$

Example 70 $A = C(X)$ *where* X *=compact Hausdroff,* $F_x(f) = f(x)$*, the default functional on* $C(X)$

Proposition 71 *Let* A *be a commutative unital Banach algebra and* $f \neq 0$ *be a multiplicative functional on* A*. Then*

1. $f(e) = 1$
2. $f(x) = 0 \ \forall x \in G(A)$
3. $f(x^{-1}) = f(x)^{-1}$

Proof. Since $f \neq 0$, so there exists at least one $a \in A$ such that $f(a) \neq 0$ and is therefore invertible. $a = ae$ so that $f(a) = f(ae) = f(a) f(e)$. Apply $f(a)^{-1}$ on both sides

Next, let $x \in G(A)$. Then, $f(xx^{-1}) = f(x) f(x^{-1}) = f(e) = 1$. Now, $f(x) = 0$, then $0 = 1$, a contradiction.

Apply $f(x)^{-1}$ on both sides to get the last answer. ■

Theorem 72 *Let* A *be a commutative unital Banach algebra and* $f \neq 0$ *be a multiplicative functional on* A*. Then* f *is continuous*

Proof. Let $f(x) = \lambda$. If $|f(x)| > \|x\|$, then $|\lambda| > \|x\|$ so that $\left\|\frac{x}{\lambda}\right\| < 1$ so that $e - \frac{x}{\lambda} \in G(A)$. Thus, $f\left(e - \frac{x}{\lambda}\right) \neq 0$ or $1 - \frac{1}{\lambda} f(x) \neq 0$ so that $f(x) \neq \lambda$. A contradiction.

Hence $|f(x)| \leq \|x\|$ so that f is bounded and any linear functional is bounded if and only if it is continuous.

Also, $\|f\| = \sup\limits_{\|x\|=1} |f(x)| \leq \sup\limits_{\|x\|=1} \|x\| = 1$

Conversely, $\|f\| = \sup\limits_{\|x\|=1} |f(x)| \geq |f(e)| = 1$

Hence $\|f\| = 1$ ■

Definition 73 *The collection of all multiplicative linear functionals is called the set of* ***characters*** *of* A *or the* ***structure space of*** A*.*

A multiplicative linear functional is also called a character. This set is denoted by $\Delta(A)$.

$\Delta(A)$ is also called the maximal ideal space, for reasons that will become clear in the next topic.

6 Ideals

Definition 74 *Let A be a commutative unital Banach Algebra. A subset I of A is said to be an ideal of A if*

1. *I is a subspace of A*
2. *If $a \in A$, $x \in I$, then $ax \in I$*

$\{0\}$ and A are improper ideals of A. All other ideals are proper ideals.

Definition 75 *A proper ideal M of A is said to be maximal if it is not properply contained in any other proper ideal of A. That is, if $M \subseteq K \subseteq A$, then either $M = K$ or $K = A$*

Theorem 76 *Let A be a commutative Banach Algebra. Then, A is a division algebra if only if it has no non-trivial ideal.*

Proof. Suppose A is a division algebra. Let $I \neq \{0\}$ be a non-zero ideal of A. We will show that $A = I$. Let $0 \neq x \in I$

$\Longrightarrow x \in A$

$\Longrightarrow$ there exists $x^{=1} \in A$ such that $xx^{-1} = x^{-1}x = e \in A$

Now, for $x \in I$ and $x^{-1} \in A$

$\Longrightarrow xx^{-1} = e \in I$

Then, for any $a \in A$, $ae = a \in I$ by definition of ideal

Hence $A = I$ and A has no non-trivial ideal

Conversely, assume that A has no non-trivial ideal. We show that A is a division algebra.

Let $0 \neq x \in A$. Consider $I = \{xy : y \in A\}$

Then, for $a, b \in I$, there exists $y_1, y_2 \in A$ such that $a = xy_1$ and $b = xy_2$. For $\alpha, \beta \in \mathbb{C}$,

$\alpha a + \beta b$

$= \alpha xy_1 + \beta xy_2$

$= x(\alpha y_1 + \beta y_2) = y$ since A is closed.

Hence $\alpha a + \beta b \in I$. Next, for $a \in A$ and $i \in I$, $i = xy_1$ and $ai = axy_1 = xay_1 = xy \in I$. Hence I is an ideal.

But since A has no non-trivial ideals, then either $I = \{0\}$ or $I = A$. If $I = \{0\}$, then $x = 0$, contradiction. Hence $I = A$. Hence $e \in I$. Thus for $x \in A$, there exists y such that $xy = e$. Since x was arbitrary, therefore A is a division algebra. ■

Theorem 77 *If A is a commutative Banach Algebra, then $\overline{I}$ is a proper ideal of A when I is a proper ideal of A.*

Proof. We first show that $\overline{I}$ is an ideal of A. Take $x, y \in \overline{I}$. Then, there exists sequences $x_n, y_n \in I$ such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$. Now, $\|x_n + y_n - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \longrightarrow 0$. Hence $x_n + y_n \longrightarrow x + y$. Next, for $\alpha \in \mathbb{C}$, $\|\alpha x_n - \alpha x\| = |\alpha| \|x_n - x\| \longrightarrow 0$. Hence $\alpha x_n \longrightarrow \alpha x$. Hence for $x, y \in \overline{I}$, we have αx and $x + y \in \overline{I}$. Thus, $\overline{I}$ is a subspace of A. Now, for $a \in A$, $\|ax - ax_n\| \leq \|a\| \|x_n - x\| \longrightarrow 0$. Hence $ax \in \overline{I}$. Therefore, $\overline{I}$ is an ideal. To show that $\overline{I}$ is proper, assume by way of contradiction that it is not. Then, $\overline{I}.= \{0\}$ or $\overline{I} = A$. In the former, we since $I \subseteq \overline{I}$, we have $I = \{0\}$, implying the contradiction that I is improper. If $\overline{I} = A$, then $e \in A$ implies $e \in \overline{I}$. Thus e is a limit point of I. By definition of limit point, for every nbd $N_\epsilon(e)$, we have $N_\epsilon(e) \cap I \neq \varnothing$. We know that $G(A)$ is open. Thus, we must also have $G(A) \cap I \neq \varnothing$. Hence there exists $x \in G(A)$ and $x \in I$. This implies there exists $x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = e$. By definition of ideal, $e \in I$. Then, for any $a \in A$, $ae = a \in I$ by definition of ideal. Hence $A = I$ implying the contradiction that I is improper.

Hence $\overline{I}$ is proper. ■

Theorem 78 *Let M be a maximal ideal in a commutative banach algebra with unity A. Then, M is closed.*

Proof. In order to show that M is closed, we show that $M = \overline{M}$. We already have $M \subseteq \bar{M}$. By previous theorem, M is proper implies $\bar{M}$ is proper. But M is maximal. Thus, $M \subseteq \bar{M} \subseteq A$ implies either $M = \bar{M}$ or $\bar{M} = A$. The latter is impossible since $\bar{M}$ is proper. Hence $M = \bar{M}$ ■

Theorem 79 *Let A be a commutative Banach Algebra. Then, no proper ideal contains any invertible element of A*

Proof. Assume by way of contradiction that a proper ideal I of A contains an invertible element. Then, $G(A) \cap I \neq \varnothing$. Hence there exists $x \in G(A)$ and $x \in I$. This implies there exists $x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = e$. By definition of ideal, $e \in I$. Then, for any $a \in A$, $ae = a \in I$ by definition of ideal. Hence $A = I$ implying the contradiction that I is improper. ■

Theorem 80 *Each proper ideal of A is contained in some maximal ideal of A*

Proof. Let I be any proper ideal of a A. Define a set $P = \{J \subset A : J \text{ is ideal and } I \subseteq J\}$. Then, $(P, \subseteq)$ is a poset

a) clearly, $J \subseteq J$ for any $J \in P$

b) If $J \subseteq K$ and $K \subseteq J$ for $J, K \in P$, then $J = K$

c) If $J \subseteq K$ and $K \subseteq L$ for $J, K, L \in P$, then $J \subseteq L$

Hence $(P, \subseteq)$ is a poset. Let $\mathcal{L}$ be a totally ordered subset of P. This means that $\mathcal{L}$ is a set of proper ideals of A which contain I and for every two elements J, K of $\mathcal{L}$ either $J \subseteq K$ or $K \subseteq L$. Let $M_{\mathcal{L}} = \bigcup_{J \in \mathcal{L}} J$. Since $M_{\mathcal{L}}$ is the union of every $J \in \mathcal{L}$, we have that $M_{\mathcal{L}} = \bigcup_{J \in \mathcal{L}} J \supseteq J$ for every $J \in \mathcal{L}$. That is, $M_{\mathcal{L}}$ is an upper bound for $\mathcal{L}$. To show that $M_{\mathcal{L}}$ is in P, we need to check that $M_{\mathcal{L}}$ is a proper ideal of A and contains I.

Since each J is a proper ideal, we have $e \notin J$ for each $J \in \mathcal{L}$. This implies that $e \notin M_{\mathcal{L}}$, hence that $M_{\mathcal{L}} \neq A$.

Next, for $a, b \in M_{\mathcal{L}}$. Then, there is some $J_a, J_b \in \mathcal{L}$ with $a \in J_a$ and $b \in J_b$. Since $\mathcal{L}$ is totally ordered, then either $J_b \subseteq J_a$ or $J_a \subseteq J_b$. If $J_a \subseteq J_b$, then $a, b \in J_b$ implies $\alpha a+\beta b \in J_b$ for any $\alpha, \beta \in \mathbb{C}$. Also, since $J_b \subseteq M_{\mathcal{L}}$, this implies that $\alpha a + \beta b \in M_{\mathcal{L}}$. Again, if $J_b \subseteq J_a$, then $a, b \in J_a$ implies $\alpha a + \beta b \in J_a$ for any $\alpha, \beta \in \mathbb{C} \Longrightarrow \alpha a + \beta b \in M_{\mathcal{L}}$. Hence $M_{\mathcal{L}}$ is a subspace.

Next, for $x \in M_{\mathcal{L}}$ and $a \in A$, we have $x \in J$ for some $J \subseteq M_{\mathcal{L}}$. Hence by definition of ideal, $xa \in J$

$\Longrightarrow ax \in M_{\mathcal{L}}$

Hence $M_{\mathcal{L}}$ is an ideal

Clearly, $I \subseteq J \subseteq M_{\mathcal{L}}$

Thus, $M_{\mathcal{L}} \in P$

Having verified the hypothesis of Zorn's lemma, we are guaranteed the existence of some maximal element $M \in P$ of P. Thus, M is a maximal ideal and $I \subseteq M$. Since I was arbitrary, this completes the proof. ■

Theorem 81 *Let A be a unital commutative Banach Algebra. Then, for any linear functional $\phi : A \longrightarrow \mathbb{C}$, if I is an ideal of A, then $\phi(I)$ is an ideal of $\mathbb{C}$*

Proof. Let I be an ideal of A. We show that $\phi(I)$.

Let $a, b \in \phi(I)$. Then, there exists $x, y \in I$ such that $a = \phi(x)$ and $b = \phi(x)$. Now, for $\alpha, \beta \in \mathbb{C}$, we have

$\alpha\phi(x) + \beta\phi(y)$

$= \phi(\alpha x + \beta y)$

Now, I is a subspace implies $\alpha x + \beta y \in I$. Hence $\phi(\alpha x + \beta y) \in \phi(I)$

or $\alpha\phi(x) + \beta\phi(y) \in \phi(I)$

or $\alpha a + \beta b \in \phi(I)$

Next, if $a \in \phi(I)$, then there exists $x \in I$ such that $\phi(x) = a \in \phi(I)$. For $z \in \mathbb{C}$, since I is a subspace and closed under scalar multiplication, we must have $zx \in I$. Hence $\phi(zx) \in \phi(I)$. Now, $za = z\phi(x) = \phi(zx)$, thus $za \in \phi(I)$

Hence $\phi(I)$ is an ideal of $\mathbb{C}$. ■

Theorem 82 *Let A be a commutative unital Banach Algebra and $f \in \Delta(A)$. Then, $\ker f$ is a maximal ideal of A*

Proof. To show that $\ker f$ is a maximal ideal, we first show that $\ker f$ is an ideal of A. Since f is a multiplifcative linear functional, so $f(0) = 0$. Thus, $0 \in \ker f$ hence $\ker f$ is non-empty. Let $x, y \in \ker f$ and $\alpha, \beta \in \mathbb{C}$. Then, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = 0$. Hence $\alpha x + \beta y \in \ker f$, implying that $\ker f$ is a subspace. For $x \in \ker f$ and $a \in A$, $f(ax) = f(a) f(x) = 0$. Hence $ax \in \ker f$

To show that $\ker f$ is maximal, assume that there exists an ideal M of A such that $\ker f \subseteq M \subseteq A$. Since M is proper, and f is linear, so by above theorem, $f(M) \subset \mathbb{C}$ is a proper ideal of $\mathbb{C}$. But the only ideals of $\mathbb{C}$ are $\{0\}$ or $\mathbb{C}$ itself so $f(M) = \{0\}$ or $f(M) = \mathbb{C}$

If $f(M) = \{0\}$, then $\ker f = M$

If $f(M) = \mathbb{C}$, then $M = A$

That is, $\ker f \subseteq M \subseteq A$ implies $\ker f = M$ or $M = A$

Hence $\ker f$ is maximal. ■

7 Quotient Algebra

Let X be a normed space and M be a closed subspace of X. We define an equivalence relation on X by $x \sim y \bmod m$ iff $y - x \in M$. Now we show that $\sim$ is an equivalence relation.

1) Since M is a subspace of X, so $0 \in M$. Hence $x - x \in M$ so that x$\sim x$

2) $x \sim y$ implies $x - y \in M$. Since M is a subspace, then it must also have additive inverses, implying that $-(x-y) = y - x \in M$. Hence $y \sim x$

3) Let $x \sim y$ and $y \sim z$. Then, $x - y \in M$ and $y - z \in M$. Since M is a subspace, it is closed under addition. Hence $x - y + y - z = x - z \in M$. Therefore, $x \sim z$

For $x \in M$, we define an equivalence class $C_x = \{y \in X : y \sim x\}$
$= \{y \in X : x - y \in M\}$
$= \{y \in X : x - y = m \text{ for some } m \in M\}$
$= \{y \in X : y = x + m \text{ for some } m \in M\}$
$= x + M$
$= [x]$

For $x \neq y$, we have $C_x \cap C_y = \varnothing$

Proof. Let $C_x \cap C_y \neq \varnothing$
Then, there exists $a \in C_x \cap C_y$
Thus, $a \sim x$ and $a \sim y$
By reflexivity, $x \sim a$ and $a \sim y$
By transitivity, $x \sim y$
That is, $x - y \in M$
or $x - y = m$ for some m
But $0 \in M$
Hence $x - y = 0$ ■

Definition 83 *The set of all equivalence classes of X is denoted and defined as* $X/M = \{C_x : x \in X\}$
$= \{x + M : x \in X\}$

Note that the set of all equivalence classes of X form a partition of X
i.e. $X = \bigcup_{x \in X} C_x$ and $C_x \cap C_y = \varnothing$ or $C_x = C_y$

Theorem 84 *If X is a Banach Algebra, M is closed ideal of X. Show that X/M is also a Banach Algebra*

Proof. Let us define $+, \cdot$ by $(x + M) + (y + M) := (x + y) + M$, $(x + M)(y + M) := xy + M$ and $\alpha(x + M) := \alpha x + M$

We show that these operations are well-defined. Let $x_1 + M = x_2 + M$. and $y_1 + M = y_2 + M$

Then, $x_1 - x_2 \in M$ and $y_1 - y_2 \in M$
That is, there exists $m_x, m_y \in M$ such that $x_1 - x_2 = m_x$ and $y_1 - y_2 = m_y$
In other words, $x_1 = m_x + x_2$ and $m_y + y_2 = y_1$
We have to show that $(x_1 + y_1) + M = (x_2 + y_2) + M$

In other words, $C_{x_1+y_1} = C_{x_2+y_2}$

Let $a \in C_{x_1+y_1}$, then $a = x_1 + y_1 + m_a$ for some $m_a \in M$

$\implies a = m_x + x_2 + m_y + y_2 + m_a$

$\implies a = x_2 + y_2 + m$ for some $m \in M$. Here $m = m_a + m_x + m_y$. Here, we have used the fact that addition is commutative.

Thus, $a \in C_{x_2+y_2}$. The converse can be proved similarly.

Now we show scalar multiplication is well-defined.

Let $x + M = y + M \iff x - y = m$ for some $m \in M$

We have to show that $\alpha x + M = \alpha y + M$

Now, $\alpha m \in M$ since M is a subspace. Hence $\alpha x - \alpha y \in M \iff \alpha x + M = \alpha y + M$

Finally, we show that multiplication is well-defined. To this end, we prove that $m\,(x + M) = M$

Let $a \in$

Let $x_1 + M = x_2 + M$. and $y_1 + M = y_2 + M$

Then, $x_1 - x_2 \in M$ and $y_1 - y_2 \in M$

That is, there exists $m_x, m_y \in M$ such that $x_1 - x_2 = m_x$ and $y_1 - y_2 = m_y$

In other words, $x_1 = m_x + x_2$ and $m_y + y_2 = y_1$

We have to show that $(x_1 y_1) + M = (x_2 y_2) + M$

Let $a \in (x_1 y_1) + M$. Then, $a = x_1 y_1 + m_1$ for some $m_1 \in M$

$a = (m_x + x_2)\,(m_y + y_2) + m_1$

$= m_x m_y + m_x y_2 + x_2 m_y + x_2 y_2 + m_1$

Since M is an ideal, $m_x m_y + m_x y_2 + x_2 m_y + m_1 \in M$

Hence $m_x m_y + m_x y_2 + x_2 m_y + m_1 = m$ for some $m \in M$

Thus, $a = x_2 y_2 + m$

$a \in C_{x_2 y_2}$

Similarly, the converse can be proved.

Thus, the operations are well-defined. We show that X/M is a Banach Algebra. First we show that it is a vector space.

1) $x + M + (y + M + z + M)$

$= x + M + (y + z + M)$

$= x + (y + z) + M$

$= (x + y) + z + M$

$= (x + y + M) + (z + M)$

$(x + M + y + M) + z + M$

2) $x + M + y + M$

$= x + y + M$

$= y + x + M$

$= y + M + x + M$

3) The additive identity is M because $(x + M) + M = M + (x + M) = x + M$

4) Let $x + M \in X/M$ be arbitrary. Then, $x \in X$

$\implies -x \in X$

$\implies (-x) + M \in X/M$. This is the additive identity for every $x + M \in X/M$

5) $\alpha\,[(x + M) + (y + M)]$

$= \alpha\,(x + y + M)$

$= \alpha x + \alpha y + M$

$= (\alpha x + M) + (\alpha y + M)$

6) $(\alpha + \beta)(x + M)$

$= [(\alpha + \beta) x] + M$

$= \alpha x + \beta x + M$

$= (\alpha x + M) + (\beta x + M)$

$= \alpha (x + M) + \beta (x + M)$

7) $\alpha (\beta (x + M))$

$= \alpha (\beta x + M)$

$= \alpha\beta x + M$

$= (\alpha\beta) x + M$

$= (\alpha\beta)(x + M)$

8) $1 (x + M)$

$= 1x + M$

$= x + M$

Next, we show that this is an algebra

1) $(x + M)(y + M) = xy + M$

Since $xy \in X$, $xy + M \in X/M$. Hence multiplication is closed.

2) $[(x + M)(y + M)](z + M)$

$= (xy + M)(z + M)$

$= (xy) z + M$

$= x (yz) + M$

$= x [(y + M)(z + M)] + M$

$= (x + M)[(y + M)(z + M)]$

3) $(x + M)[(y + M) + (z + M)]$

$= (x + M)[y + z + M]$

$= x (y + z) + M$

$= xy + xz + M$

$= (xy + M) + (xz + M)$

$= [(x + M)(y + M)] + [(x + M)(z + M)]$

4) $\alpha [(x + M)(y + M)]$

$= \alpha (xy + M)$

$= \alpha (xy) + M$

$= (\alpha x) y + M$

$= (\alpha x + M)(y + M)$

$= [\alpha (x + M)](y + M)$

The third last line is also equal to $x (\alpha y) + M$

$= (x + M)((\alpha y) + M)$

$= (x + M)[\alpha (y + M)]$

Now, we prove that this is a Normed space. We define $\|.\| : X/M \longrightarrow \mathbb{R}$ by $\|x + M\|_{X/M} = \inf_{m \in M} \|x - m\|$. To show that this is a norm,

1) Since for $x \in X$, $m \in M \subset X$, $x - m \in X$. Hence $\|x - m\| \geq 0$

$\Longrightarrow \inf_{m \in M} \|x - m\| \geq 0$

$\Longrightarrow \|x + M\|_{X/M} \geq 0$

Also, $\|x + M\|_{X/M} = 0$

$\iff \inf_{m\in M} \|x - m\| = 0$

That is, the distance from x to M is zero. But M is closed. Hence $\inf_{m\in M} \|x - m\| = 0$

$\iff x \in M$

$\iff x + M = M$. This is the zero of X/M

2) Let $\alpha \in \mathbb{C}$

Then, $\|\alpha (x + M)\|_{X/M}$

$= \inf_{m\in M} \|\alpha x - \alpha m\|$

$= \inf_{m\in M} |\alpha| \|x - m\|$

$= |\alpha| \inf_{m\in M} \|x - m\|$

$= |\alpha| \|x + M\|_{X/M}$

3) Let $x + M, y + M \in X/M$. Then,

$\|x + M + y + M\|_{X/M}$

$= \|x + y + M\|_{X/M}$

$= \inf_{m\in M} \|x + y - m\|$

$= \inf_{m\in M} \left\|x - \frac{1}{2}m + y - \frac{1}{2}m\right\|$

$\leq \inf_{m\in M} \left\|x - \frac{1}{2}m\right\| + \inf_{m\in M} \left\|y - \frac{1}{2}m\right\|$

$= \|x + M\|_{X/M} + \|y + M\|_{X/M}$

To show that this is a Normed Algebra,

$\|(x + M)(y + M)\|_{X/M}$

$= \|xy + M\|_{X/M}$

$= \inf_{m\in M} \|xy - m\|$

$\leq \inf_{m\in M} \|xy - xw - my + wm\|$

$= \inf_{m\in M} \|x (y - w) - m (y - w)\|$

$= \inf_{m\in M} \|(x - m)(y - w)\|$

$\leq \inf_{m\in M} \|(x - m)\| \|(y - w)\|$

$= \inf_{m\in M} \|x - m\| \inf_{w\in M} \|y - m\|$

$\|x + M\|_{X/M} \|y + M\|_{X/M}$

For completeness, let $a_n = \sum x_n + M$ be an absolutely convergent series in X/M. Then,

$\sum \|x_n + M\| < \infty$

But $\|x_n + M\| = \inf_{m\in M} \|x_n - m\|$.

By definition of infimum, for each n, there exists $v_n \in M$ such that $\|x_n - v_n\| \leq \|x_n + M\| + \frac{1}{2^n}$

Now, $\sum \|x_n - v_n\| \leq \sum \left(\|x_n + M\| + \frac{1}{2^n}\right)$

$= \sum \|x_n + M\| + \sum \frac{1}{2^n} < \infty$

Thus, $\sum \|x_n - v_n\| < \infty$

That is, $\sum x_n - v_n$ converges absolutely in the Banach Algebra X. Thus, it converges in the ordinary sense.

Let $\sum x_n - v_n = x \in X$. Then, $\left\| \left[\sum_{n=1}^{N} x_n + M \right] - (x + M) \right\| = \left\| \sum_{n=1}^{N} (x_n - x) + M \right\|$

$\leq \left\| \sum_{n=1}^{N} (x_n - x) - v_n \right\|$

$= \left\| \sum_{n=1}^{N} (x_n - v_n) - x \right\| \longrightarrow 0$ as $N \longrightarrow \infty$

That is, $\left\| \left[\sum_{n=1}^{N} x_n + M \right] - (x + M) \right\| \longrightarrow 0$ as $N \longrightarrow \infty$

So that the sequence of partial sums of the series a_n converges. Hence X/M is complete ■

Theorem 85 *An ideal M of a commutative unital Banach Algebra A is maximal if and only if A/M is a division algebra*

Proof. Let M be a maximal ideal of A. Consider the canonical map $\pi : A \longrightarrow A/M$ defined by $\pi(x) = x + M$. For $x = y$, we clearly have $x + M = y + M$ hence π is well-defined.

We show that $\pi \in \Delta(A)$

$\pi(x + y) = x + y + M$

$= x + M + y + M = \pi(x) + \pi(y)$

Next, $\pi(\alpha x) = \alpha x + M = a(x + M) = \alpha\pi(x)$

Finally, $\pi(xy) = xy + M = (x + M)(y + M) = \pi(x)\pi(y)$

Since M is maximal, M is a proper subspace of A so there exists $x \in A \backslash M$. Consider $J = \{ax + y : a \in A, y \in M\}$

Now, $0 \in A$ and $0 \in M$. Hence $0x + 0 = 0$ implies $0 \in J$. Next, let $p = a_1x + y_1$ and $q = a_2x + y_2$ belong to J. Then, $p - q = (a_1 - a_2)x + (y_1 - y_2) \in J$. Furthermore, for $\alpha \in A$, $\alpha ax + \alpha y \in J$. Thus, J is an ideal of A. Since, $e \in A$, $0 \in M$ for $ex + 0 = x \in J$ implies $x \notin M$. . Hence $M \subset J$. By maximality of M, we have $J = A$. Since $e \in A$, we have $ax_0 + y_0 = e$ and, therefore, $\pi(ax + y) = \pi(e)$

$\implies \pi(a)\pi(x) + \pi(y) = \pi(e)$

Since $y \in M$, $\pi(y) = 0$. Thus, $\pi(a)\pi(x) = \pi(e)$. That is, for any arbitrary element $\pi(x) \in A/M$., we have an inverse $\pi(a)$

Conversely, suppose that A/M is a division algebra. We have to show that M is a maximal ideal of A. For this, let us suppose that A is not maximal. Then, either $M = A$ or there exists an ideal J of A such that $M \subset J \subset A$.

If $M = A$, then $A/M = \{0\}$. That is, A/M has no non-zero element. This contradicts the fact that A/M is a division algebra.

If $M \subset J \subset A$, then $J/M \subset A/M$

And J/M is an ideal of A/M. So that $J/M \neq \{0\}$ but this contradicts the fact that A/M is a division algebra since a division algebra has no maximal ideals. ■

Theorem 86 *If M is a maximal ideal of A, then there exists $f \in \Delta(A)$ such that* $\ker f = M$

Proof. Since M is a maximal ideal of A, then A/M is a division algebra. By the Gelfand-Mazur theorem, A/M is isometrically ismorphic to $\mathbb{C}$.

Denote this by $\Psi : A/M \longrightarrow \mathbb{C}$. Define $\pi : A \longrightarrow A/M$ by $\pi(x) = x + M$.

Let $f = \Psi \circ \pi$. Then, $f(a+b) = (\Psi \circ \pi)(a+b)$

$= \Psi(\pi(a) + \pi(b)) = \Psi(\pi(a)) + \Psi(\pi(b)) = f(a) + f(b)$

Furthermore, $f(\alpha a) = \Psi(\pi(\alpha a)) = \Psi(\alpha \pi(a)) = \alpha \Psi(\pi(a)) = \alpha f(a)$

Finally, $f(ab) = \Psi(\pi(ab)) = \Psi(\pi(a)\pi(b)) = \Psi(\pi(a))\Psi(\pi(b))$

Hence $f \in \Delta(A)$. Now, $\ker f = \{x \in A : f(x) = 0\}$

or $\ker f = \{x \in A : \Psi \circ \pi(x) = 0\}$

or $\ker f = \left\{x \in A : \pi(x) = \widehat{0}\right\}$ since Ψ is one-one

or $\ker f = \{x \in A : \pi(x) = M\}$

or $\ker f = \{x \in A : x + M = M\}$

or $\ker f = \{x \in A : x \in M\}$

That is, $\ker f = M$ ■

Theorem 87 *Let A be a commutative Banach unital Algebra and $f, g \in \Delta(A)$ such that $\ker f = \ker g = M$ where M is maximal. Then, $f = g$*

Proof. As M is maximal, there exists $x_0 \in A \backslash M$. Then, $x_0 \in \ker g$. Hence $g(x_0) \neq 0$. Let $\beta = \frac{g(x)}{g(x_0)}$ and let $m = x - \beta x_0$ for $m, x \in A$ Then, $g(m) = g(x) - \beta g(x_0)$

$\implies g(x) - \frac{g(x)}{g(x_0)} g(x_0) = 0$. Hence $m \in \ker g = M$

For $f(m) = f(x) - \beta f(x_0)$

We have $f(m) = 0$ and, therefore, $f(x) = \beta f(x_0)$

$\implies f(x) = \frac{g(x)}{g(x_0)} f(x_0)$

or $f(x) = \alpha g(x)$ where $\alpha = \frac{f(x_0)}{g(x_0)}$. This holds for all x. Hence $f = \alpha g$

Now, $\alpha [g(x_0)]^2 = \alpha g(x_0^2) = f(x_0^2) = \alpha^2 [g(x_0)]^2$

Hence $\alpha = \alpha^2$. This is only valid if $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$, then f, g are trivial and $\ker f = \ker g = M = A$, implying the contradiction that M is improper. Thus, $\alpha = 1$ and $f = g$ ■

Thus, there is a one-one correspondence between the set of maximal ideals of A and $\Delta(A)$, justifying the name "maximal ideal space".

Theorem 88 *Let A be a unital, commutative Banach Algebra and let $x \in A$. Then, $\lambda \in \sigma(x)$ if and only if $\lambda = f(x)$ for some $f \in \Delta(A)$*

Proof. Let $\lambda \in \sigma(x)$. Then, $x - \lambda e$ is not invertible. Since for any invertible element x, $f(x) \neq 0$ so, there exists at least one $f \in \Delta(A)$ such that $f(x - \lambda e) = 0$. Or $f(x) = \lambda$

Conversely, suppose that $f(x) = \lambda$ for some $f \in \Delta(A)$. Then, $f(x - \lambda e) = 0$ or $x - \lambda e$ is not invertible. Hence $\lambda \in \sigma(x)$ ■

Exercise 89 *Show that there is only one multiplicative linear functional on* $\mathbb{C}$*, which is* $f(z) = z$

Solution 90 *Let* $f \in \Delta(\mathbb{C})$*. Then,* $f(1) = 1$ *implies* $f(q) = qf(1) = q$ *for* $q \in \mathbb{Q}$*. That is* $f|_{\mathbb{Q}} = I$*. Now for any* $f, g \in \Delta(\mathbb{C})$*,* f, g *are identity on* $\mathbb{Q}$ *and* f, g *are continuous. Let* x *be an irrational limit for a sequence of rational terms* x_n*. Then,* $f(x_n) = g(x_n)$*. By continuity, we must have* $f(x) = g(x)$ *after the application of limits on both sides. Since* x *was arbitrary, therefore* $f|_{\mathbb{R}} = I = g|_{\mathbb{R}}$*. Finally, since* $f(i)^2 = f(i^2) = f(-1) = -1$ *and since the only roots of* $x^2 = -1$ *are* $\pm i$*, then* $f(i) = i$ *or* $f(i) = -i$*. That is, for any* $f, g \in \Delta(\mathbb{C})$*,* $f(z) = z$ *or* $f(z) = \overline{z}$ *But conjugation is not a multiplicative linear functional. Hence* $f(z) = z$ *is the only multiplicative linear functional*

Solution 91 *Let* $f \in \Delta(A)$*. Then,* $f(\alpha z) = \alpha f(z)$*. But this scalar is also a vector. Hence* $f(\alpha z) = f(\alpha) f(z)$

That is, $\alpha f(z) = f(\alpha)$ *or* $f(\alpha) = \alpha$ *for any* $\alpha \in \mathbb{C}$ *and* $f(z) \neq 0$ *for* $z \neq 0$*. Since the invertible elements of* $\mathbb{C}$ *and hence those elements for which* $f(z) \in G(\mathbb{C})$ *is* $\mathbb{C}\setminus\{0\}$*, it follows that* $\ker f = \{0\}$*. Hence* $f(z) = z$ *is the only multiplicative linear functional.*

8 Weak Convergence

We know that in calculus one defines different types of convergnce. We've seen such types: ordinary convergence, absolute convergence and uniform convergence. We now move on to consider a weaker version of convergence but in order to justify the word "weak", we will call our usual understanding of convergence as strong convergence. More specifically,

Definition 92 *A sequence* (x_n) *in a normed space* X *is said to be* ***strongly convergent*** *if there is an* $x \in X$ *such that* $\lim_{n\to\infty} \|x_n - x\| = 0$

Again, this will be shortened to $x_n \longrightarrow x$ or $\lim_{n\to\infty} x_n = x$. x will be called a strong limit.

Weak converge provides a sense in which a sequence is convergent based on some particular support.

Definition 93 *A sequence* (x_n) *in a normed space* X *is said to be* ***weakly convergent*** *if there is an* $x \in X$ *such that for every* $f \in X'$ $\lim_{n\to\infty} |f(x_n) - f(x)| = 0$.

This will be written $x_n \xrightarrow{w} x$. In a sense, we are mapping each member of a sequence to a natural or real number, depending on the underlying field. That is, we have a sequence $(a_n) = (f(x_n))$. This allows us to resort to the familiar theorems specific for real and complex numbers.

Theorem 94 *Let* $x_n \xrightarrow{w} x$*. Then,*

1. *The weak limit x of (x_n) is unique*

2. *Every subsequence of (x_n) converges weakly to x*

3. *The sequence $\|x_n\|$ is bounded*

Proof. 1. Suppose $x_n \xrightarrow{w} x$ and $x_n \xrightarrow{w} y$. Then, $f(x_n) \longrightarrow f(x)$ and $f(x_n) \longrightarrow f(y)$. Since $f(x_n)$ is a sequence of real or complex numbers, its limit is unique. That is, $f(x) = f(y)$

$\Longrightarrow f(x-y) = 0$ for all f

Hence $x = y$

2. This follows from the fact that if a real or complex sequence is convergent, then every subsequence converges to the same limit as the sequence

3. Since $(f(x_n))$ is convergent, it is bounded, say $|f(x_n)| \leq c_f$ for all n, where c_f depends on f but not on n. Define $g_{x_n}(f) = f(x_n)$. Then, $g_{x_n}(f)$ is bounded for every $f \in X'$. Since X' is complete regardless of the completion of X, we can apply the uniform boundedness theorem to X'' and get $\|g_{x_n}\|$ bounded. By another corollary, $\|x_n\| = \|g_{x_n}\|$ ■

Finite dimensional spaces make life easier; here's another reason why:

Theorem 95 *In a finite dimensional space, strong convergence and weak convergence are equivalent*

Proof. First we show that strong convergence implies weak convergence with the same limit. If $x_n \longrightarrow x$. Then, for any $f \in X'$ $|f(x_n) - f(x)| \leq \|f\| \|x_n - x\| \longrightarrow 0$ hence $x_n \xrightarrow{w} x$

Conversely, suppose $x_n \xrightarrow{w} x$ and $\dim X = k$. Then, $x_n = \alpha_1^{(n)} e_1 + ... + \alpha_k^{(n)} e_k$ and $x = \alpha_1 e_1 + ... + \alpha_k e_k$. By assumption, $f(x_n) \longrightarrow f(x)$ for any f. We take in particular $f_1, ..., f_k$ defined by $f_j(e_k) = \delta_{jk}$. Then, $f_j(x_n) = \alpha_j^{(n)}$ and $f_j(x) = \alpha_j$ hence $f_j(x_n) \longrightarrow f_j(x)$. From this, we readily obtain $\|x_n - x\| = \left\| \sum_{j=1}^{k} \left(\alpha_j^{(n)} - \alpha_j\right) e_j \right\| \leq \sum_{j=1}^{k} \left|\alpha_j^{(n)} - \alpha_j\right| \|e_j\| \longrightarrow 0$ hence $x_n \longrightarrow x$ ■

As might have been guessed, there are infinite dimensional spaces where a sequence may converge weakly but not strongly:

Take an orthonormal sequence (e_n) in a Hilbert Space H. Since every $f \in H'$ has a Riesz representation, $f(x) = \langle x, z \rangle$. Hence $f(e_n) = \langle e_n, z \rangle$. From the Bessel inequality, $\sum_{j=1}^{\infty} |\langle e_n, z \rangle|^2 \leq \|z\|^2$ so that the series on the left converges to zero. That is, $\langle e_n, z \rangle = f(e_n) \longrightarrow 0$. Since f in arbitrary, we see that $e_n \longrightarrow 0$ but that is true since $\|e_n - e_m\|^2 = \langle e_m - e_n, e_m - e_n \rangle = 2$

Exercise 96 *If $x_n \in C[a,b]$ and $x_n \xrightarrow{w} x \in C[a,b]$, show that (x_n) is pointwise convergent on $[a,b]$*

Solution 97 *We have to show that* $x_n(t) \longrightarrow x(t)$ *for every* $t \in [a,b]$. *Functionals* f_{t_0} *of* $C[a,b]$ *are defined for vectors* $x(t) \in C[a,b]$ *such that* $f_{t_0}(x(t)) = x(t_0)$ *for* $t_0 \in [a,b]$. *Hence, for any sequence of functions (vectors)* $x_n(t)$ *in* $C[a,b]$, $x_n(t) \xrightarrow{w} x(t)$
$\Longrightarrow f_{t_0}(x_n(t)) \longrightarrow f_{t_0}(x(t))$
$\Longrightarrow x_n(t_0) \longrightarrow x(t_0)$ *for any* $t_0 \in C[a,b]$. *Hence weak convergence implies point-wise convergence in* $C[a,b]$

Exercise 98 *Let* X *and* Y *be normed spaces.,* $T \in B(X,Y)$ *and* (x_n) *a sequence in* X. *If* $x_n \xrightarrow{w} x_0$, *show that* $T(x_n) \xrightarrow{w} T(x_0)$

Solution 99 *Let* $x_n \xrightarrow{w} x_0$. *Then,* $|f(x_n) - f(x)| \longrightarrow 0$. *From* $\|f\| = \sup\limits_{0 \neq x \in X} \frac{|f(x)|}{\|x\|}$, *we have* $\|x\| = \sup\limits_{0 \neq f \in X'} \frac{|f(x)|}{\|f\|}$. *Thus,* $\|x_n - x_0\| = \sup\limits_{0 \neq f \in X'} \frac{|f(x-x_0)|}{\|f\|}$ *so that for any* $g \in Y'$ *and for any* $T \in B(X,Y)$, *we have* $|g(T(x_n)) - g(T(x))| = |g(T(x_n) - T(x))| = |g(T(x_n - x))|$
$\leq \|g\| \|T(x_n - x)\|$
$\leq \|g\| \|T\| \|x_n - x\|$
$= \|g\| \|T\| \sup\limits_{0 \neq f \in X'} \frac{|f(x-x_0)|}{\|f\|} \longrightarrow 0$

Weak convergence covers scalar multiplication and vector addition.

Lemma 100 *If* (x_n) *and* (y_n) *are sequences in the same normed space* X, *show that* $x_n \xrightarrow{w} x$ *and* $y_n \xrightarrow{w} y$ *implies* $x_n + y_n \xrightarrow{w} x + y$ *as well as* $\alpha x_n \xrightarrow{w} \alpha x$

Proof. Let $x_n \xrightarrow{w} x$ and $y_n \xrightarrow{w} y$. Then, for all $\epsilon > 0$, we have N_1 such that $|f(x_n) - f(x)| < \epsilon/2 \; \forall n \geq N_1$ and N_2 such that $|g(y_n) - g(y)| < \epsilon/2 \; \forall n \geq N_2$ for all $g, f \in X^{'}$. Let $N = \max\{N_1, N_2\}$ and choose the particular $f = g$. Then, $|f(x_n + y_n) - f(x - y)|$
$= |f(x_n) - f(x) + f(y_n) - f(y)|$
$= |f(x_n) - f(x) + g(y_n) - g(y)| \leq |f(x_n) - f(x)| + |g(y_n) - g(y)| < \epsilon$ for all $n \geq N$
$\Longrightarrow f(x_n + y_n) \longrightarrow f(x + y)$
$\Longrightarrow x_n + y_n \xrightarrow{w} x + yy - x$
Similarly, we can have $|f(x_n) - f(x)| < \epsilon / |\alpha|$ and $|f(\alpha x_n) - f(\alpha x)|$
$= |\alpha f(x_n) - \alpha f(x)|$
$= |\alpha| |f(x_n) - f(x)| < \epsilon$
$\Longrightarrow \alpha x_n \xrightarrow{w} \alpha x$ ∎

Exercise 101 *Show that* $x_n \xrightarrow{w} x_0$ *implies* $\lim\limits_{n \to \infty} \inf \|x_n\| \geq \|x_0\|$

Solution 102 *For any weakly convergent sequence* $x_n \xrightarrow{w} x_0 \neq 0$*, we can choose* n_k *such that the subsequence* $\|x_{n_k}\| \longrightarrow \lim_{n\to\infty} \inf \|x_n\|$*. Note that this does not violate the fact that every subsequence converges weakly to* x_0*. Now, by Hahn-Banach theorem, there exists* $f \in X^{'}$ *such that* $\|f\| = 1$ *and* $f(x_0) = \|x_0\|$*. Then,* $|f(x_{n_k})| \leq \|f\| \|x_{n_k}\| = \|x_{n_k}\|$ *and*

$\Longrightarrow \lim_{n_k\to\infty} |f(x_{n_k})| \leq \lim_{n_k\to\infty} \|x_{n_k}\|$

$\Longrightarrow \left| \lim_{n_k\to\infty} f(x_{n_k}) \right| \leq \lim_{n\to\infty} \inf \|x_n\|$

$\Longrightarrow \left| f\left(\lim_{n_k\to\infty} x_{n_k} \right) \right| \leq \lim_{n\to\infty} \inf \|x_n\|$

$\Longrightarrow |f(x_0)| \leq \lim_{n\to\infty} \inf \|x_n\|$ *since every subsequence converges weakly to the same limit*

$\Longrightarrow \|x_0\| \leq \lim_{n\to\infty} \inf \|x_n\|$

Exercise 103 ***If*** $x_n \xrightarrow{w} x_0$ *in a normed space* X*, show that* $x_0 \in \bar{Y}$ *where* $Y = span(x_n)$

Solution 104 *Assume that* $x_0 \notin \overline{Y} \Longrightarrow x_0 \in X - \overline{Y}$*. Then, the conditons satisfy the statement of theorem 4.6-7. Hence there exists* $f \in X^{'}$ *such that* $|f(y)| = 0$ *for all* $y \in \overline{Y}$ *and* $f(x_0) = \delta = \inf_{y\in\overline{Y}} \|y - x_0\|$

Since $Y = span(x_n)$*, then* $x_n \in Y \Longrightarrow f(x_n) = 0$ *for all* n*. Hence* $f(x_n) \longrightarrow f(x_0)$ *implies* $f(x_0) = 0 = \inf_{y\in\overline{Y}} \|y - x_0\| \Longrightarrow x_0 \in \overline{Y}$*. Contradiction.*

Exercise 105 ***If*** (x_n) *is a weakly convergent sequence, show that there is a sequence* (y_m) *of linear combinations of elements of* (x_n) *which converges strongly to* x_0

Solution 106 *From the previous exercise, we have that any element* y_m *of* Y *is a linear combination of* (x_n)*. Since* $x_0 \in \overline{Y}$*. therefore either* x_0 *is a limit point or it belongs to* Y*. In the first case* $x_n \longrightarrow x_0$ *strongly. If* x_0 *is not a limit point, then it belongs to* Y *and is, therefore, a linear combination of* (x_n)*, in which case for any linear functional,* $f(x_0) = f\left(\sum \alpha_{n_k} x_{n_k}\right)$ *implying divergence of the sequence* $f(x_n)$*, which is a contradiction.*

Corollary 107 *Any closed subspace* Y *of a normed space* X *contains the limits of all weakly convergent sequences of elements.*

Definition 108 *A **weak Cauchy sequence** in a real or complex normed space* X *is a sequence* (x_n) *in* X *such that for every* $f \in X'$*, the sequence* $(f(x_n))$ *is Cauchy in* $\mathbb{R}$ *or* $\mathbb{C}$*.*

Note that $\lim_{n\to\infty} f(x_n)$ exists. A weak Cauchy sequence is bounded

Proof. Let x_n be a weak Cauchy sequence. Then, for any given $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $|f(x_n) - f(x_m)| < \epsilon$ for $n, m \geq N$. Choose $b = \max\{f(x_1), f(x_2), ..., f(x_{N-1}), \epsilon\}$. Then, $|f(x_n)| \leq b$ for all n. ■

Furthermore, every non-empty subset containing a weak Cauchy sequence is bounded

Proof. Let A be a set in a normed space X such that every nonempty subset of A contains a weak Cauchy sequence. Assume that A is not bounded. Then, there exists an unbounded sequence in A such that $\|x_n\| \longrightarrow \infty$. Since every subsequence converges to the same limit, we can find a weak Cauchy subsequence which is unbounded, a contradiction. ■

Definition 109 *A normed space X is said to be* ***weakly complete*** *if each weak Cauchy sequence in X converges weakly in X.*

Lemma 110 *If X is reflexive, then X is weakly complete.*

Proof. If a normed space is reflexive, then it is complete. It remains to prove that every complete space is weakly complete. This follows from the fact that strong convergence implies weak convergence. ■

9 Weak Topology

Definition 111 *Let X be a set and let τ be a family of subsets of X. τ is called a* ***topology*** *on X if*

1. *$X, \varnothing \in \tau$*
2. *Any union of elements of τ is an element of τ*
3. *Any intersection of finitely many elements of τ is an element of τ*

Definition 112 *A subset Z of X is called τ-open or simply* ***open*** *if $Z \in \tau$. A is called* ***closed*** *if $Z^c \in \tau$. For $x \in X$, $N \subseteq X$ is called* ***neighbourhood*** *of x if there exists $U \in \tau_X$ such that $x \in U \subseteq N$. A collection $\mathcal{B}$ of subsets of X is called a* ***basis*** *if a) $\forall x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B$ and b) if $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then $\exists B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$*

A direct consequence of this is as follows: $\tau = \left\{ \bigcup_i B_i : B_i \in \mathcal{B} \text{ and } i \in I \right\}$. $\mathcal{C} = \{U : \forall U \in \tau \text{ and } \forall x \in U, \exists V : x \in V \subset U\}$ is a basis. $\mathcal{S} = \left\{ A : A \subseteq X \text{ and } \bigcup A = X \right\}$ is called a **subbasis**. In this case, $\tau = \left\{ \bigcup B : B = \bigcap_{i=1}^{n} S_i \text{ for } S_i \in \mathcal{S} \right\}$. Thus, a subbasis is formed by taking finite intersections of elements of the basis and, therefore $\mathcal{S} \subset \mathcal{B}$.

Definition 113 *Let* (X, τ_X) *and* (Y, τ_Y) *be two topological spaces. Then,* $f : (X, \tau_X) \longrightarrow (Y, \tau_Y)$ *is called* ***continuous*** *if for every* $V \in \tau_Y$*, we have* $f^{-1}(V) \in \tau_X$

Definition 114 *Let* $\{X_i : i \in I\}$ *be a family of topological spaces. Consider*

$$X = \prod_{i \in I} X_i$$

Then, X *is called* ***product topology*** *or* ***Tychonoff Topology*** *if it has the fewest open sets for which all projections projection* $p_i : X \longrightarrow X_i$ *are continuous.*

Theorem 115 (Tychonoff Theorem) *Assuming the Axiom of Choice, the product of a family of compact space is compact in the product topology.*

Definition 116 *Let* τ_1 *and* τ_2 *be two topologies on a set* X*, and assume* $\tau_1 \subset \tau_2$*; that is, every* τ_1*-open set is also* τ_2*-open. Then we say that* τ_1 *is* ***weaker*** *than* τ_2*, or that* τ_2 *is* ***stronger*** *than* τ_1*.*

An equivalent way of saying this is as follows: $\mathcal{B}_i$ is a basis for τ_i, then $\mathcal{B}_2 \subset \mathcal{B}_1$. Furthermore, in this situation, the identity mapping on X is continuous from (X, τ_2) to (X, τ_1).

Definition 117 *Let* (X, τ_X) *be a topological space.* X *is called* ***Hausdroff*** *if for all* $x, y \in X$*, there exists a neighbourhood* U *of* x *and a neighbourhood* V *of* y *such that* $U \cap V = \varnothing$*.*

Definition 118 *Let* A *be an indexing set. The set* $C = \{U_\alpha : \alpha \in A\}$ *of indexed family of sets* U_α *is a* ***cover*** *of* X *if* $X \subseteq \bigcup_{\alpha \in A} U_\alpha$*. A topological space is called* ***compact*** *if every open cover can be reduced to a finite subcover.*

It follows that every finite set is compact.

Proposition 119 *Compact sets in Hausdroff Spaces are closed*

Proof. This is easy to see in the finite case. For the infinite case, let C be a compact subset of a Hausdroff Space X. Then, for every open cover, we can have a finite subcover of neighbourhoods $U(x_i)$ for $1 \leq i \leq n$. X being Hausdroff implies that $V(x_i)$ for $n+1 \leq i \leq m$ is disjoint from $U(x_i)$ for each i. Consider $V_a = \bigcap V(x_i)$ for some $a \in X$. Then, $V(x) \subseteq X - C$ and, therefore, $X - C = \bigcup_{a \in X} V_a$ is open. ■

Not every compact set is closed. For instance, in the cofinite topology (a topology in which every closed set is finite) for $\mathbb{Z}$, every subset of $\mathbb{Z}$ is compact, some of which are closed (finite) and some of which (infinite ones) are open. To see this, consider two cases of a subset A. If A is finite, then it is compact. If it isn't, then one can always find a finite subcover by simply taking union of two open infinite sets, which can always be found, given the definition of cofinite.

Proposition 120 *Every closed subset of a compact set is compact*

Proof. Let (X, τ_X) be a compact topological space and let $A^c \in \tau_X$. Let C be an open cover of A^c. Then, $A \cup C$ is also an open cover of A. Since (X, τ_X) is compact, therefore C can be reduced to a finite subcover $\mathcal{C} = \{U_1, U_2, ..., U_k\}$. Since $X \subseteq \bigcup \mathcal{C}$ and $A \subseteq X$, therefore $A \subseteq \bigcup \mathcal{C}$, which implies that A is compact by definition. ■

Proposition 121 *Countable product of Hausdroff spaces is Hausdroff*

Proof. Let $\{(X_\alpha, \tau_\alpha)\}$ be a family of Hausdroff spaces for $\alpha \in I$. Consider $(X, \tau) = \prod_{a \in I} (X_\alpha, \tau_\alpha)$

(a_α), (b_α) be two disjoint points of (X, τ). Then, $a_i \neq b_i$ for some $i \in I$. Since (X_i, τ_i) is Hausdroff so there exist two disjoint open sets $U_i, V_i \subseteq S_i$ such that $a_i \in U_i$, $b_i \in V_i$. In this case, $\prod_{a=1}^{i-1} (X_\alpha, \tau_\alpha) \times U_i \times \prod_{a=i+1} (X_\alpha, \tau_\alpha)$ and $\prod_{a=1}^{i-1} (X_\alpha, \tau_\alpha) \times V_i \times \prod_{a=i+1} (X_\alpha, \tau_\alpha)$ are two open disjoint sets in the product space (X, τ) containing the original points. ■

Proposition 122 *$\tau_1 \subset \tau_2$ are topologies on a set X, if τ_1 is a Hausdorff topology, and if τ_2 is compact, then $\tau_1 = \tau_2$*

Proof. Let $F^c \in \tau_2$. That is, F is τ_2-closed. Since X is τ_2-compact, and F is a closed subset, therefore F is τ_2-compact. From, $\tau_1 \subset \tau_2$ we have $F^c \in \tau_1$. Also, every opencover of τ_1 is also an open cover of τ_2. Therefore, F is τ_1-compact. Since τ_1 is a Hausdorff topology, it follows from the previous proposition that F is τ_1-closed. That is, $F^c \in \tau_1$. Thus, $\tau_2 \subset \tau_1$ ■

We know that a norm space gives rise to a metric space and hence a topological space. We also know that equivalent norms on N define the same topology for N.

Proof sketch. Suppose that a norm $\|.\|_1$ generates a topology τ_1 and $\|.\|_2$ generates the topology τ_2. What this means is that open sets (balls) in either topology will be formed from their respective norms. To prove that the topologies are equivalent, we need to prove that they are both subsets of each other. This can be done by showing that every open set in one topology is contained in another because of the radius of the ball is less than or equal to than a constant times the radius of the other. ■

We may define a possibly different (weaker) topology on N using the continuous dual space N^*. Obviously for any $f \in N^*$, f is linear and bounded and, therefore, continuous. By defining a new topology on N, we must ensure that the continuity of functionals is not compromised. We, therefore, want a topology on N which should be the coarsest topology with respect to N^* such that each element of N^* remains continuous. Rigorously,

Definition 123 *Given a Banach Algebra A over a topological field $\mathbb{F}$, the* ***weak topology*** τ_w *on A is the coarsest topology on A such that each $f : (A, \tau_w) \longrightarrow \mathbb{F}$ is continuous.*

By continuity, we must have open sets as the inverse images of open sets. Thus, it makes sense to require that the initial topology be described as the topology generated by sets of the form $f^{-1}(V)$, where V is an open set in $\mathbb{F}$. This makes sure that open sets of A are formed in the smallest possible way with respect to $\mathbb{F}$. Now, since the arbitrary union and finite intersection of open sets of a topological space are open, we can use $f_i^{-1}(V)$ to generate for ourselves a topology, which is the weak topology. It, therefore, makes sense to have a subbasis. A subbasis for the weak topology is the collection of sets of the form $f^{-1}(V)$ where $f \in A^*$ and V is an open subset of $\mathbb{F}$.

It is easy to see that we can have $\bigcup\limits_i f^{-1}(V_i) = A$ for some $i \in I$

In the topological setting, convergence is defined as follows: a sequence $x_n \longrightarrow x \in X$ if $\forall V \in \tau$ containing x, there exists a natural number N such that $x_n \in V$ for all $n \geq N$. Similarly, a sequence x_n is Cauchy if $\forall U \in \tau$ containing 0, there exists a natural number N such that $x_n - x_m \in U$ for all $n, m \geq N$. With this in mind, we assert that the consideration of a weak topology coincides with the idea of weak convergence. Specifically, a sequence (x_n) in A converges in the weak topology τ to the element x of A if and only if $f(x_n) \longrightarrow f(x)$ for all $f \in A^*$.

Proof. ($\Longrightarrow$) Since $f \in A^*$ is continuous, this side is trivial.

($\Longleftarrow$)Let $x \in U \in \tau$. By definition, U is made by union of $\bigcap\limits_{i=1}^{n} f^{-1}(V_i)$ for all $f \in A^*$. Thus, $f(x) \in V_i$ for all i. By hypothesis, $f(x_n) \longrightarrow f(x)$. That is, $\forall V_i \in \tau$ containing $f(x)$, there exists a natural number N_i such that $f(x_n) \in V$ for all $n \geq N$. Letting $N = \max N_i$ completes the proof. ■

Let E be a bounded subset of a Banach Algebra A. The collection of sets $\mathcal{B} = \{U_{\epsilon,E} : \epsilon > 0, E \subset A\}$, where $U_{\epsilon,E} = \{f \in A^* : |f(x)| < \epsilon \forall x \in E\}$ is a neighbourhood of the zero functional, defines a basis for a topology. Notice that $\hat{0} \in A^*$ and that for $f = \hat{0}$, $|f(x)| < \epsilon$ for all $x \in E \subset A$. Hence the defined $U_{\epsilon,E}$ are non-empty and, therefore, any such two neighbourhoods have a nonempty intersection. Note that for $E = \varnothing$, $U_{\epsilon,E} = \varnothing$.

Now we show that such a collection forms a basis. That is, a) Every $f \in A^*$ is contained in some $U_{\epsilon,E}$ and b) If $x \in U, V \in \Omega$, then $\exists W \in \Omega$ such that $x \in W \subset U \cap V$

a)

For any $x \in A$, define $E = \{x\}$. Then, E is bounded. Thus, for any $f \in A^*$, $|f(x)| \leq \|f\| \|x\| < \|f\| \|x\| + \delta = \epsilon$ for $\delta > 0$. Hence for any $f \in A^*$ and $x \in A$, we can have $U_{\epsilon,E}$ containing f where $E = \{x\}$ and $\epsilon = \|f\| \|x\| + \delta$.

b)

Let $x \in U_{\epsilon_1,E_1}, U_{\epsilon_2,E_2}$. That is, there exists $f_1, f_2 \in A^*$ such that $|f_1(x)| < \epsilon_1$ and $|f_2(y)| < \epsilon_2$ for all $x \in E_1$ and $y \in E_2$. Define $E := E_1 \cap E_2 \subset E_i$ where $i = 1, 2$ and E is bounded since E_i's are bounded. There are two cases to consider. If $A_1 \cap A_2$ is empty, then (b) is vacuously true. If $E_1 \cap E_2 \neq \varnothing$, then define $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then, $|f(x)| < \epsilon$, $\forall x \in E$ where $f \in U_{\epsilon_1,E_1} \cap U_{\epsilon_2,E_2}$. This intersection is always non-empty. Hence we have shown that there exists a set $U_{\epsilon,E}$ such that $U_{\epsilon,E} \subseteq U_{\epsilon_1,E_1} \cap U_{\epsilon_2,E_2}$

Since such a collection forms a basis, it suffices to say that the collection of arbitrary unions of $U_{\epsilon,E}$ with varying ϵ, E forms a topology. However, we prove this directly.

Proof. $U_{\epsilon,E}$ is open. We show that $U^c_{\epsilon,E} = \{f \in A^* : |f(x)| \geq \epsilon \forall x \in E\}$ is closed. Let f be a limit point of $U^c_{\epsilon,E}$. Then, there exists a sequence f_n in $U^c_{\epsilon,E}$ such that $f_n \longrightarrow f$. That is, $|f_n(x)| \geq \epsilon$ for all $x \in E$. By continuity of $|.|$, $|f(x)| \geq \epsilon$. We also show that f is linear. Since every ϵ' neighbourhood of f intersects $U_{\epsilon,E}$, we must have $|f(x) - f_n(x)| < \epsilon'/3$, $|f(y) - f_n(y)| < \epsilon'/3$ and $|f(x+y) - f_n(x+y)| < \epsilon'/3$ for $x, y \in E$. Note that $f_n(x+y) = f_n(x) + f_n(y)$ for all n hence $|f(x+y) - f(x) - f(y)| = |f(x+y) + f_n(x+y) - f_n(x) - f_n(y) - f(x) - f(y)|$

$\leq |f(x) - f_n(x)| + |f(y) - f_n(y)| + |f(x+y) - f_n(x+y)| < \epsilon'$

Hence $f(x+y) = f(x) + f(y)$

Next, let $|f_n(\alpha x) - f(\alpha x)| < \epsilon'/2$ and $|f(x) - f_n(x)| < \epsilon'/2|\alpha|$. Hence $|f(\alpha x) - \alpha f(x)| = |f(\alpha x) - \alpha f(x) + f_n(\alpha x) - \alpha f_n(x)|$

$\leq |f(\alpha x) - f_n(\alpha x)| + |\alpha f(x) - \alpha f_n(x)| < \epsilon'$. Thus, $f(\alpha x) = \alpha f(x)$

Thus, each $U_{\epsilon,E}$ is a member of the weak topology. By the same reason and by the construction of a set $U_{\epsilon,E}$ such that $U_{\epsilon,E} \subseteq U_{\epsilon_1,E_1} \cap U_{\epsilon_2,E_2}$, the intersection of two open sets is open.

Now, let U_{ϵ_i,E_i} be elements of basis $\mathcal{B}$ for some indexing set I. For $\epsilon = \max_i \{\epsilon_i\}$ and $E = \bigcup_i E_i$, we have $U_{\epsilon,E} = \bigcup_i U_{\epsilon_i,E_i}$. To obtain $\varnothing$, we take the empty union. Thus, arbitrary union of open sets is open. It is easy to see that we can generate the dual space itself since every functional has a non-emtpy kernel.

Thus, the neighbourhoods of zero generate a topology for the dual space. ■

As a remark, we mention that every finite set is bounded. Thus, $\mathcal{B} = \{U_{\epsilon,E} : \epsilon > 0, E \subset A\} \supset \mathcal{A} = \{U_{\epsilon,E} : \epsilon > 0, C \subset A\}$ where C is a finite set, E is a bounded set. Thus, $\tau_{\mathcal{A}} \supset \tau_{\mathcal{B}}$.

We now move on to consider a topology on the original Banach Space. Consider $\mathcal{C} = \left\{S(x_0, \epsilon, C^*) : \epsilon > 0, C^* \subset A'\right\}$ where C^* is a finite set, and $S(x_0, \epsilon, C^*) = \{x \in A : |f(x) - f(x_0)| < \epsilon \forall f \in C^*\}$. We first prove that $\mathcal{C}$ is a basis.

Proof. a) Since f is continuous, for every $\epsilon > 0$, we can find a δ such that $|f(x) - f(x_0)| < \epsilon$ whenever $\|x - x_0\| < \delta$. That is, given any $\epsilon > 0$, we can find $x \in A$ in a δ-nbd of x_0. Hence (a) holds. By the same reason, $S(x_0, \epsilon, C^*)$ is open.

b) Now, let $C_1^* = \{f_1, f_2, ..., f_n\}$ and let $C_2^* = \{g_1, g_2, ..., g_m\}$. Consider $S(x_0, \epsilon_1, C_1^*)$ and $S(x_0, \epsilon_2, C_2^*)$. Again, $x \in C^* := C_1^* \cap C_2^*$ is either empty or non-empty but finite in both cases. In the former, there is nothing left to prove. In the latter, let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $C^* = \{h_1, h_2, ...h_k\}$ where $k \leq \min\{m, n\}$ to prove (b). The existence of C^* is, therefore, independent of the linear dependence of elements from C_1^* and C_2^*. ■

Thus, $\mathcal{C}$ generates a topology. If we want subbasis, we need to consider finite intersections of $S(x_0, \epsilon, C^*)$. These open sets are some of the inverse images spoken of in the definition of subbasis for the topology on A.

Definition 124 *The coarsest topology on A^* with respect to which all functionals F_x in A^{**} are continuous is called* ***weak-∗ topology***

We now consider the neighbourhood

Problem 125 *Show that the collection of sets $\Omega = \{S(f_0, \epsilon, E) : \epsilon > 0, E \subset A\}$ where A is a Banach Algebra, E is a bounded set and $S(f_0, \epsilon, E) = \{f_0 \in A^* : |F_x(f) - F_x(f_0)| < \epsilon \forall x \in E\} = \{f \in A^* : |f(x) - f_0(x)| < \epsilon \forall x \in E\}$ defines a basis for a topology*

Solution 126 *a) Let $f \in A^*$. Since $|F_x(f) - F_x(f_0)| = |f(x) - f_0(x)| \leq \|f - f_0\| \|x\| < \epsilon$ for all $x \in E$, hence (a) holds*

b) Consider $S(f_0, \epsilon_1, E_1) \cap S(g_0, \epsilon_2, E_2)$. Then, define $E := E_1 \cap E_2$ and $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. If E is empty, then so is $S(f_0, \epsilon_1, E_1) \cap S(g_0, \epsilon_2, E_2)$. If not, then clearly, $S(h_0, \epsilon, E) \subset S(f_0, \epsilon_1, E_1) \cap S(g_0, \epsilon_2, E_2)$

Now, if the collection of sets $\Omega_1 = \{S(f_0, \epsilon, E) : \epsilon > 0, E \subset A\}$ where E is a finite set defines a basis for a topology (which it does), then $\Omega_1 \subset \Omega$ and hence $\tau_\Omega \subset \tau_{\Omega_1}$.

Note that the weak-∗ topology is generated by $\|f\|_x = |f(x)|$ and also that for any $F_x : A^* \longrightarrow \mathbb{C}$ is continuous.

Theorem 127 *The weak-∗ topology is Hausdroff*

Proof. Take two functions f and g. That is, we must have a vector $x \in A$ such that $f(x) \neq g(x)$. We have to show that their neighbourhoods are disjoint. Take $\epsilon = \frac{|f(x)-g(x)|}{3} > 0$. If $S(x, f, \epsilon) \cap S(x, g, \epsilon) \neq \varnothing$, then let $h \in S(x, f, \epsilon) \cap S(x, g, \epsilon)$

In this case, $|h(x) - f(x)| < \epsilon$ and $|h(x) - g(x)| < \epsilon$. Now, for $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| < 2\epsilon$

Hence $|f(x) - g(x)| < 2.\frac{|f(x)-g(x)|}{3}$ which implies $3 < 2$, a contradiction. Hence $S(x, f, \epsilon) \cap S(x, g, \epsilon) = \varnothing$ ■

Theorem 128 (Banach-Algalou Theorem) *Unit ball is weak-∗ compact.*

Proof. Take $K = \{f \in A^* : \|f\| \leq 1\}$. With each vector $x \in A$, associate a compact, Hausdroff space C_x where C_x is the closed ball $\{z : z \leq \|x\|\}$. By the Tychonoff theorem, the product $C = \prod C_x$ is also compact. Since each For each x, the value of $f(x)$ for all $f \in K$ lies in C_x since $|f(x)| \leq \|f\| \|x\| \leq \|x\|$ so we can consider $K \subseteq C$ where $f \in K$ is associated with $(f(x_1), f(x_2), ...) \in C$ for $x_1, x_2, ... \in A$. In fact, we have embedded K in C. Since the closed subset of compact set is compact in the Hausdroff topology, to show that K is compact, we show that K is closed. Let $h \in K^d$. Then, there exist $h_n \in K$ such that $h_n \longrightarrow h$. $h_n \in K$ implies $\|h_n\| \leq 1$. By continuity of norm, $\|h\| \leq 1$. This h is also linear. ■

Theorem 129 *The structure space of a commutative Banach Algebra A is weak-$*$ closed.*

Proof. For any multiplicative linear functional f, $\|f\| = 1$, as proved. Hence $\Delta(A) \subseteq K = \{f \in A^* : \|f\| \leq 1\}$. Hence $\overline{\Delta(A)}.\subseteq K$. Let Let $h \in \overline{\Delta(A)}$. Then, h is linear. Therefore in order to show that $h \in \Delta(A)$, all we have to show is $h(xy) = h(x)h(y)$ for all $x, y \in A$

Consider the weak-$*$ neighbourhood $W = W(X, h, \epsilon)$ of h for $X = \{x, y, x+y, xy\}$. Since $f(xy) - f(x) - f(y) = 0$ for $f \in W$ we have $|h(xy) - h(x)h(y)| \leq |h(xy) - f(xy)| + |h(x) - f(x)| + |f(y) - h(y)| \longrightarrow 0$. ■

Corollary 130 *$\Delta(A)$ is weak-$*$ compact.*

Proof. This follows since K is compact and Hausdroff and $\Delta(A)$ is closed. ■

9.1 Gelfand Topology

The **Gelfand topology** of $\Delta(A)$ is the weak-$*$ topology induced by (A^*, τ_w). That is, $\left(\Delta(A), \tau_{\Delta(A)}\right) = \{Y \cap X : X \in \tau_w\}$.

For each $x \in A$, define a mapping $\hat{x} : \Delta(A) \longrightarrow \mathbb{C}$ by $\hat{x}(f) = f(x)$ for all $f \in \Delta(A)$. $\hat{x} : \Delta(A) \longrightarrow \mathbb{C}$ is continuous by definition of weak-$*$ topology.

Consider the mapping $\varphi : A \longrightarrow \hat{A}$ where $\hat{A} \subset A^{**}$ such that $D(\hat{x}) = \Delta(A)$ for all $x \in A$ where $\varphi(x) = \hat{x}$. This is well-defined

Proof. $x = y$

$\Longrightarrow f(x) = f(y)$

$\Longrightarrow \hat{x}(f) = \hat{y}(f)$ for all $f \in \Delta(A)$

$\Longrightarrow \hat{x} = \hat{y}$

$\Longrightarrow \varphi(x) = \varphi(y)$ ■

φ is called the **Gelfand transform**.

Definition 131 *Let A be a unital Banach Algebra and let $\mathcal{M}$ be the set of all maximal ideals of A. The **radical** or **Jacobson radical** of A, denoted by rad$A$, is the intersection of all maximal ideasl of A.*

That is, $\text{rad}A = \bigcap_{M \in \mathcal{M}} M$.

We have seen before that there is a one-one correspondence between the elements of $\Delta(A)$ and the set $\mathcal{M}$ of all maximal ideals of A, therefore $\text{rad}A = \bigcap_{f \in A^*} \ker f$.

A is called semisimple if $\text{rad}A = \{0\}$. Note that rad$A$ is a two-sided ideal of A. We have seen that each maximal ideal is closed.

Example 132 *Let $A = \mathbb{C}^n$ with coordinatewise algebraic operations. Maximal ideals of $\mathbb{C}^n$ are $M_i = \{(x_1, x_2, ..., x_n) : x_j \in \mathbb{C},\ x_i = 0,\ 1 \leq j \leq n\}$. rad$A = \{0\}$ hence A is semisimple. Projection operations P_i are corresponding multiplicative linear functions with $\ker P_i = M_i$*

Example 133 *Let* $A = C(X)$*. For each* $x \in X$*,* $M_x = \{f \in C(X) : f(x) = 0\}$ *are maximal ideals. Clearly,* A *is semisimple.*

Theorem 134 $\Delta(A)$ *is weak-*$*$ *compact and Hausdroff*

Proof. $\Delta(A)$ is a weak-$*$ closed subset of A^* which is a compact, Hausdroff space. Hence $\Delta(A)$ is weak-$*$ compact. $\Delta(A)$ is weak-$*$ Hausdroff since Hausdroff property is hereditary. We prove this directly. $\left(\Delta(A), \tau_{\Delta(A)}\right) = \{Y \cap X : X \in \tau_w\}$ where for any distinct $x, y \in A^*$, there exist disjoint neighbourhoods $U, V \in \tau_w$ such that $x \in U$ and $y \in V$. For distinct $f, g \in \Delta(A) \implies f, g \in A^*$ and, therefore, exist disjoint neighbourhoods $U, V \in \tau_w$ such that $f \in U$ and $g \in V$. Thus, $f \in \Delta(A) \cap U$ and $g \in \Delta(A) \cap V$ such that $(\Delta(A) \cap U) \cap (\Delta(A) \cap V)$

$= \Delta(A) \cap (U \cap V) = \varnothing$. By definition, $\Delta(A) \cap V, \Delta(A) \cap U \in \tau_{\Delta(A)}$ ■

Theorem 135 *The Gelfand representation* $x \longmapsto \hat{x}$ *is a homomorphism of* A *onto* $\hat{A}$ *whose kernel is the radical of* A*. Furthermore, the Gelfand representation is an isomorphism iff* A *is semisimple*

Proof. Note that $\hat{A} \subset C(\Delta(A))$ is a subalgebra. The algebraic operations on $\hat{A}$ are defined pointwise. Let $\hat{x}, \hat{y} \in \hat{A}$ and $\alpha \in \mathbb{C}$. That is, $(\hat{x} + \hat{y})(f) = \hat{x}(f) + \hat{y}(f)$, $(\alpha\hat{x})(f) = \alpha(\hat{x}(f))$ and $(\hat{x}\hat{y})(f) = [\hat{x}(f)][\hat{y}(f)]$

We have to prove that

(i) $\varphi(x + y) = \varphi(x) + \varphi(y)$

(ii) $\varphi(\alpha x) = \alpha\varphi(x)$

(iii) $\varphi(xy) = \varphi(x)\varphi(y)$

(i) $\varphi(x + y) = \widehat{x + y} = f(x + y)$

$= f(x) + f(y) = \hat{x} + \hat{y} = \varphi(x) + \varphi(y)$

Hence (i) holds.

(ii) $\widehat{\alpha x} = f(\alpha x) = \alpha f(x) = \alpha\hat{x}$

(iii) $\widehat{xy} = f(xy) = f(x) f(y) = \hat{x}\hat{y}$

Now, $\ker\varphi = \{x \in A : \varphi(x) = 0\}$

$= \{x \in A : \hat{x} = 0\}$

$= \{x \in A : \hat{x}(f) = 0 \forall f\}$

$= \bigcap_{f \in A^*} \ker f = \text{rad} A$

φ is an isomorphism

$\iff \ker\varphi = \{0\} = \text{rad} A$

$\iff A$ is semisimple ■